1

Vectors and Linear Spaces

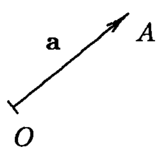
Vectors provide a mathematical formulation for the notion of direction, thus making direction a part of our mathematical language for describing the physical world. This leads to useful applications in physics and engineering, notably in connection with forces, velocities of motion, and electrical fields. Vectors help us to visualize physical quantities by providing a geometrical interpretation. They also simplify computations by bringing algebra to bear on geometry.

1.1 Scalars and vectors

In geometry and physics and their engineering applications we use two kinds of quantities, scalars and vectors. A **scalar** is a quantity that is determined by its magnitude, measured in units on a suitable scale. [1] For instance, mass, temperature and voltage are scalars.

A **vector** is a quantity that is determined by its direction as well as its magnitude; thus it is a *directed quantity* or a *directed line-segment*. For instance, force, velocity and magnetic intensity are vectors.

We denote vectors by boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{r}$, etc. [or indicate them by arrows, $\vec{a}, \vec{b}, \vec{r}$, etc., especially in dimension 3]. A vector can be depicted by an arrow, a line-segment with a distinguished end point. The two end points are called the initial point (tail) and the terminal point (tip):



1. length (of the line-segment OA)
2. direction
   – attitude (of the line OA)
   – orientation (from O to A)

The length of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$. Two vectors are equal if and only

1 In this chapter scalars are real numbers (elements of $\mathbb{R}$).

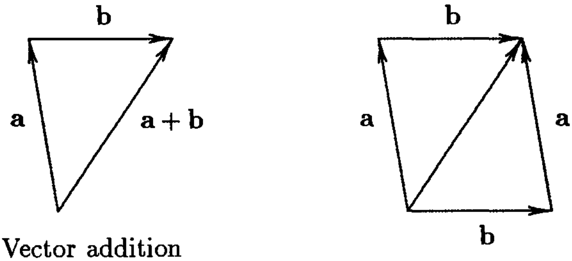
if they have the same length and the same direction. Thus,

$$\mathbf{a} = \mathbf{b} \iff |\mathbf{a}| = |\mathbf{b}| \quad \text{and} \quad \mathbf{a} \uparrow\uparrow \mathbf{b}.$$

Two vectors have the same direction, if they are parallel as lines (the same attitude) and similarly aimed (the same orientation). The *zero vector* has length zero, and its direction is unspecified. A *unit vector* $\mathbf{u}$ has length one, $|\mathbf{u}| = 1$. A vector $\mathbf{a}$ and its *opposite* $-\mathbf{a}$ are of equal length and parallel, but have opposite orientations.

1.2 Vector addition and subtraction

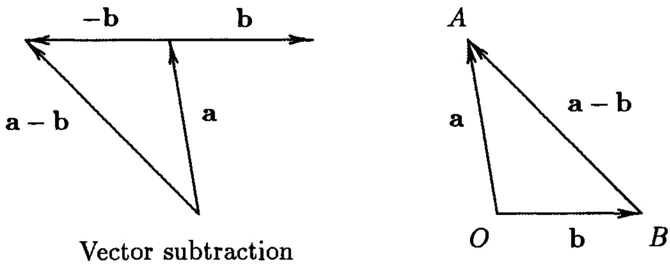
Given two vectors $\mathbf{a}$ and $\mathbf{b}$, translate the initial point of $\mathbf{b}$ to the terminal point of $\mathbf{a}$ (without rotating $\mathbf{b}$). Then the sum $\mathbf{a}+\mathbf{b}$ is a vector drawn from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$. Vector addition can be visualized by the triangle formed by vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}+\mathbf{b}$.



Vector addition

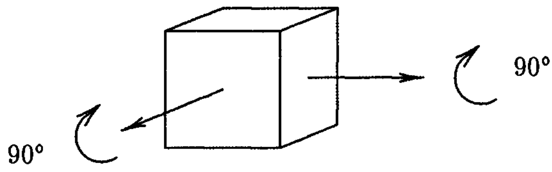
Vector addition is commutative, $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$, as can be seen by inspection of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as sides. It is also associative, $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$, and such that two opposite vectors cancel each other, $\mathbf{a}+(-\mathbf{a}) = 0$.

Instead of $\mathbf{a}+(-\mathbf{b})$ we simply write the difference as $\mathbf{a}-\mathbf{b}$. Note the order in $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ when $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.



Vector subtraction

Remark. To qualify as vectors, quantities must have more than just direction and magnitude – they must also satisfy certain rules of combination. For instance, a rotation can be characterized by a direction $\mathbf{a}$, the axis of rotation, and a magnitude $\alpha = |\mathbf{a}|$, the angle of rotation, but rotations are not vectors because their composition fails to satisfy the commutative rule of vector addition, $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$. The lack of commutativity of the composition of rotations can be verified by turning a box around two of its horizontal axes by $90°$:



The terminal attitude of the box depends on the order of operations. The axis of the composite rotation is not even horizontal, so that neither $\mathbf{a}+\mathbf{b}$ nor $\mathbf{b}+\mathbf{a}$ can represent the composite rotation. We conclude that rotation angles are not vectors – they are a different kind of directed quantities. ∎

1.3 Multiplication by numbers (scalars)

Instead of $\mathbf{a}+\mathbf{a}$ we write $2\mathbf{a}$, etc., and agree that $(-1)\mathbf{a} = -\mathbf{a}$, the opposite of $\mathbf{a}$. This suggests the following definition for multiplication of vectors $\mathbf{a}$ by real numbers $\lambda \in \mathbb{R}$: the vector $\lambda\mathbf{a}$ has length $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ and direction given by (for $\mathbf{a} \neq 0$)

$$\begin{aligned} \lambda\mathbf{a} \uparrow\uparrow \mathbf{a} \quad &\text{if} \quad \lambda > 0, \\ \lambda\mathbf{a} \uparrow\downarrow \mathbf{a} \quad &\text{if} \quad \lambda < 0. \end{aligned}$$

Numbers multiplying vectors are called *scalars*. Multiplication by scalars, or *scalar multiplication*, satisfies distributivity, $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$, $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$, associativity, $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$, and the unit property, $1\mathbf{a} = \mathbf{a}$, for all real numbers λ, μ and vectors $\mathbf{a}, \mathbf{b}$.

1.4 Bases and coordinates

In the plane any two non-parallel vectors $\mathbf{e}_1, \mathbf{e}_2$ form a *basis* so that an arbitrary vector in the plane can be uniquely expressed as a linear combination $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$. The numbers a_1, a_2 are called *coordinates* or *components* of the vector $\mathbf{a}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

When a basis has been chosen, vectors can be expressed in terms of the coordinates alone, for instance,

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1), \quad \mathbf{a} = (a_1, a_2).$$

If we single out a distinguished point, the origin O, we can use vectors to label the points A by $\mathbf{a} = \overrightarrow{OA}$. In the *coordinate system* fixed by O and $\{\mathbf{e}_1, \mathbf{e}_2\}$ we can denote points and vectors in a similar manner,

$$\text{point} \quad A = (a_1, a_2), \qquad \text{vector} \quad \mathbf{a} = (a_1, a_2),$$

since all the vectors have a common initial point O.

In coordinate form vector addition and multiplication by scalars are just coordinate-wise operations:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$
$$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2).$$

Conversely, we may start from the set $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$, and equip it with component-wise addition and multiplication by scalars. This construction introduces a real *linear structure* on the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ making it a 2-dimensional real *linear space* $\mathbb{R}^2$. The real linear structure allows us to view the set $\mathbb{R}^2$ intuitively as a plane, the *vector plane* $\mathbb{R}^2$. The two unit points on the axes give the *standard basis*

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1)$$

of the 2-dimensional linear space $\mathbb{R}^2$.

In our ordinary space a basis is formed by three non-zero vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ which are not in the same plane. An arbitrary vector $\mathbf{a}$ can be uniquely represented as a linear combination of the basis vectors:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

The numbers a_1, a_2, a_3 are coordinates [2] in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Conversely, coordinate-wise addition and scalar multiplication make the set

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

a 3-dimensional real *linear space* or *vector space* $\mathbb{R}^3$. In a coordinate system fixed by the origin O and a standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ a point $P = (x, y, z)$ and its *position vector*

$$\overrightarrow{OP} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

have the same coordinates. [3]

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2 Some authors speak about components of vectors and coordinates of points.

3 Since a vector beginning at the origin is completely determined by its endpoints, we will sometimes refer to the *point* $\mathbf{r}$ rather than to the *endpoint of the vector* $\mathbf{r}$.

1.5 Linear spaces and linear functions

Above we introduced vectors by visualizing them without specifying the grounds of our study. In an axiomatic approach, one starts with a set whose elements satisfy certain characteristic rules. Vectors then become elements of a mathematical object called a linear space or a vector space V. In a linear space vectors can be added to each other but not multiplied by each other. Instead, vectors are multiplied by numbers, in this context called scalars. [4]

Formally, we begin with a set V and the field of real numbers $\mathbb{R}$. We associate with each pair of elements $\mathbf{a}, \mathbf{b} \in V$ a unique element in V, called the *sum* and denoted by $\mathbf{a}+\mathbf{b}$, and to each $\mathbf{a} \in V$ and each real number $\lambda \in \mathbb{R}$ we associate a unique element in V, called the *scalar multiple* and denoted by $\lambda\mathbf{a}$. The set V is called a **linear space** V over $\mathbb{R}$ if the usual rules of addition are satisfied for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$

$$
\begin{array}{ll}
\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} & \text{commutativity} \\
(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) & \text{associativity} \\
\mathbf{a}+0=\mathbf{a} & \text{zero-vector } 0 \\
\mathbf{a}+(-\mathbf{a})=0 & \text{opposite vector } -\mathbf{a}
\end{array}
$$

and if the scalar multiplication satisfies

$$
\begin{array}{ll}
\left.\begin{array}{l}\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b} \\ (\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}\end{array}\right\} & \text{distributivity} \\
(\lambda\mu)\mathbf{a}=\lambda(\mu\mathbf{a}) & \text{associativity} \\
1\mathbf{a}=\mathbf{a} & \text{unit property}
\end{array}
$$

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in V$. The elements of V are called *vectors*, and the linear space V is also called a vector space. The above axioms of a linear space set up a real *linear structure* on V.

A subset U of a linear space V is called a linear *subspace* of V if it is closed under the operations of a linear space:

$$
\begin{array}{lll}
\mathbf{a}+\mathbf{b} \in U & \text{for} & \mathbf{a}, \mathbf{b} \in U, \\
\lambda\mathbf{a} \in U & \text{for} & \lambda \in \mathbb{R},\ \mathbf{a} \in U.
\end{array}
$$

For instance, $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

A function $L : U \to V$ between two linear spaces U and V is said to be *linear* if for any $\mathbf{a}, \mathbf{b} \in U$ and $\lambda \in \mathbb{R}$,

$$
\begin{array}{l}
L(\mathbf{a}+\mathbf{b})=L(\mathbf{a})+L(\mathbf{b}) \quad \text{and} \\
L(\lambda\mathbf{a})=\lambda L(\mathbf{a}).
\end{array}
$$

4 Vectors are not scalars, and scalars are not vectors. Vectors belong to a linear space V, and scalars belong to a field $\mathbb{F}$. In this chapter $\mathbb{F}=\mathbb{R}$.

Linear functions preserve the linear structure. A linear function $V \to V$ is called a linear transformation or an *endomorphism*. An invertible linear function $U \to V$ is a *linear isomorphism*, denoted by $U \simeq V$. [5]

The set of linear functions $U \to V$ is itself a linear space. A composition of linear functions is also a linear function. The set of linear transformations $V \to V$ is a ring denoted by $\mathrm{End}(V)$. Since the endomorphism ring $\mathrm{End}(V)$ is also a linear space over $\mathbb{R}$, it is an associative algebra over $\mathbb{R}$, denoted by $\mathrm{End}_{\mathbb{R}}(V)$. [6]

1.6 Linear independence; dimension

A vector $\mathbf{b} \in V$ is said to be a *linear combination* of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ if it can be written as a sum of multiples of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$, that is,

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k \quad \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}.$$

A set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is said to be *linearly independent* if none of the vectors can be written as a linear combination of the other vectors. In other words, a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is linearly independent if $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$ is the only set of real numbers satisfying

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k = 0.$$

In a linear combination

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k$$

of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are unique; we call them the *coordinates* of $\mathbf{b}$.

Linear combinations of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \subset V$ form a subspace of V; we say that this subspace is *spanned* by $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$. A linearly independent set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \subset V$ which spans V is said to be a *basis* of V. All the bases for V have the same number of elements called the *dimension* of V.

Quadratic Structures

Concepts such as *distance* or *angle* are *not* inherent in the concept of a linear structure alone. For instance, it is meaningless to say that two lines in the linear space $\mathbb{R}^2$ meet each other at right angles, or that there is a basis of

5 Finite-dimensional real linear spaces are isomorphic if they are of the same dimension.

6 A ring R is a set with the usual addition and an associative multiplication $R \times R \to R$ which is distributive with respect to the addition. An algebra A is a linear space with a bilinear product $A \times A \to A$.

equally long vectors $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{R}^2$. The linear structure allows comparison of lengths of parallel vectors, but it does not enable comparison of lengths of non-parallel vectors. For this, an extra structure is needed, namely the metric or quadratic structure.

The quadratic structure on a linear space $\mathbb{R}^n$ brings along an algebra which makes it possible to calculate with geometric objects. In the rest of this chapter we shall study such a geometric algebra associated with the Euclidean plane $\mathbb{R}^2$.

1.7 Scalar product

We will associate with two vectors a real number, the *scalar product* $\mathbf{a}\cdot\mathbf{b} \in \mathbb{R}$ of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. This scalar valued product of $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ is defined as

in coordinates	$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$
geometrically	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\varphi$

where φ $[0 \leq \varphi \leq 180°]$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The geometrical construction depends on the prior introduction of lengths and angles. Instead, the coordinate approach can be used to define the length

$$\lvert\mathbf{a}\rvert = \sqrt{\mathbf{a}\cdot\mathbf{a}},$$

which equals $\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2}$, and the angle given by

$$\cos\varphi = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *orthogonal*, if $\mathbf{a}\cdot\mathbf{b} = 0$. A vector of length one, $\lvert\mathbf{a}\rvert = 1$, is called a *unit vector*. For instance, the standard basis vectors $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$ are orthogonal unit vectors, and so form an *orthonormal basis* for $\mathbb{R}^2$.

The scalar product can be characterized by its properties:

$$\left.\begin{aligned} &(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c} \\ &(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b}) \end{aligned}\right\} \quad \text{linear in the first factor}$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \qquad \text{symmetric}$$

$$\mathbf{a}\cdot\mathbf{a} > 0 \quad \text{for} \quad \mathbf{a} \neq 0 \qquad \text{positive definite.}$$

Symmetry and linearity with respect to the first factor together imply bilinearity, that is, linearity with respect to both factors. The real linear space $\mathbb{R}^2$ endowed with a bilinear, symmetric and positive definite product is called a *Euclidean plane* $\mathbb{R}^2$.

All Euclidean planes are isometric [7] to the one with the metric/norm

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}| = \sqrt{x^2 + y^2}.$$

In the rest of this chapter we assume this metric structure on our vector plane $\mathbb{R}^2$.

Remark. The quadratic form $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}|^2 = x^2 + y^2$ enables us to compare lengths of non-parallel line-segments. The linear structure by itself allows only comparison of parallel line-segments. ∎

1.8 The Clifford product of vectors; the bivector

It would be useful to have a multiplication of vectors satisfying the same axioms as the multiplication of real numbers – distributivity, associativity and commutativity – and require that the norm is preserved in multiplication, $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$. Since this is impossible in dimensions $n \geq 3$, we will settle for distributivity and associativity, but drop commutativity. However, we will attach a geometrical meaning to the lack of commutativity.

Take two orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the vector plane $\mathbb{R}^2$. The length of the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ is $|\mathbf{r}| = \sqrt{x^2 + y^2}$. If the vector $\mathbf{r}$ is multiplied by itself, $\mathbf{rr} = \mathbf{r}^2$, [8] a natural choice is to require that the product equals the square of the length of $\mathbf{r}$,

$$\mathbf{r}^2 = |\mathbf{r}|^2.$$

In coordinate form, we introduce a product for vectors in such a way that

$$(x\mathbf{e}_1 + y\mathbf{e}_2)^2 = x^2 + y^2.$$

Use the distributive rule without assuming commutativity to obtain

$$x^2\mathbf{e}_1^2 + y^2\mathbf{e}_2^2 + xy(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) = x^2 + y^2.$$

This is satisfied if the orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ obey the multiplication rules

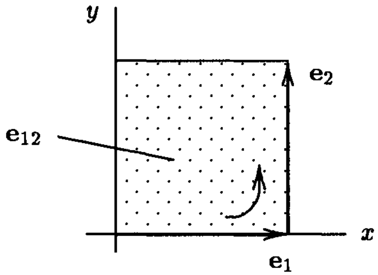
$$\boxed{\begin{aligned}\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1\\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1\end{aligned}} \quad \text{which correspond to} \quad \boxed{\begin{aligned}|\mathbf{e}_1| = |\mathbf{e}_2| = 1\\ \mathbf{e}_1 \perp \mathbf{e}_2\end{aligned}}$$

Use associativity to calculate the square $(\mathbf{e}_1\mathbf{e}_2)^2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1$. Since the square of the product $\mathbf{e}_1\mathbf{e}_2$ is negative, it follows that $\mathbf{e}_1\mathbf{e}_2$ is neither a scalar

7 An isometry of quadratic forms is a linear function $f : V \to V'$ such that $Q'(f(\mathbf{a})) = Q(\mathbf{a})$ for all $\mathbf{a} \in V$.

8 The scalar product $\mathbf{a} \cdot \mathbf{b}$ is not the same as the Clifford product $\mathbf{ab}$. Instead, the two products are related by $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$.

nor a vector. The product is a new kind of unit, called a **bivector**, representing the oriented plane area of the square with sides $\mathbf{e}_1$ and $\mathbf{e}_2$. Write for short $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$.



We define the *Clifford product* of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ to be $\mathbf{ab} = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\mathbf{e}_{12}$, a sum of a scalar and a bivector.

1.9 The Clifford algebra $C\ell_2$

The four elements

1	scalar
$\mathbf{e}_1$, $\mathbf{e}_2$	vectors
$\mathbf{e}_{12}$	bivector

form a basis for the **Clifford algebra $C\ell_2$** [9] of the vector plane $\mathbb{R}^2$, that is, an arbitrary element

$$u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12} \quad \text{in} \quad C\ell_2$$

is a linear combination of a scalar u_0, a vector $u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ and a bivector $u_{12}\mathbf{e}_{12}$. [10]

Example. Compute $\mathbf{e}_1\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2$, $\mathbf{e}_{12}\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1^2\mathbf{e}_2 = -\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_{12} = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2^2 = -\mathbf{e}_1$ and $\mathbf{e}_{12}\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2^2 = \mathbf{e}_1$. Note in particular that $\mathbf{e}_{12}$ anticommutes with both $\mathbf{e}_1$ and $\mathbf{e}_2$. ▮

The Clifford algebra $C\ell_2$ is a 4-dimensional real linear space with basis elements

9 These algebras were invented by William Kingdon Clifford (1845-1879). The first announcement of the result was issued in a talk in 1876, which was published posthumously in 1882. The first publication of the invention came out in another paper in 1878.

10 The Clifford algebra $C\ell_n$ of $\mathbb{R}^n$ contains 0-vectors (or scalars), 1-vectors (or just vectors), 2-vectors, ..., n-vectors. The aggregates of k-vectors give the linear space $C\ell_n$ a multivector structure $C\ell_n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \ldots \oplus \bigwedge^n \mathbb{R}^n$.

1, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_{12}$ which have the multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

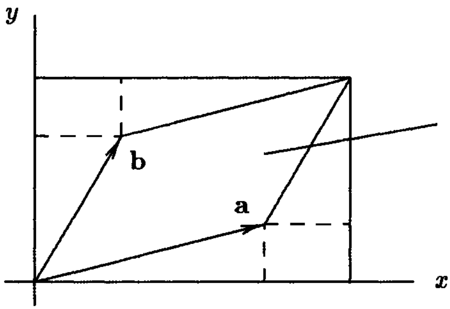
1.10 Exterior product = bivector part of the Clifford product

Extracting the scalar and bivector parts of the Clifford product we have as products of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$

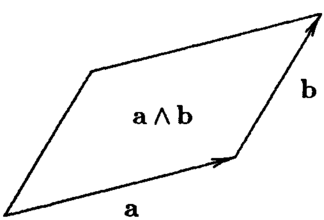
$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2,$ the scalar product '**a** dot **b**',

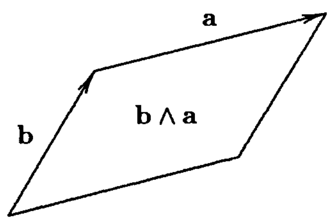
$\mathbf{a} \wedge \mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12},$ the exterior product '**a** wedge **b**'.

The bivector $\mathbf{a} \wedge \mathbf{b}$ represents the oriented plane segment of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. The area of this parallelogram is $|a_1b_2 - a_2b_1|$, and we will take the *magnitude* of the bivector $\mathbf{a} \wedge \mathbf{b}$ to be this area $|\mathbf{a} \wedge \mathbf{b}| = |a_1b_2 - a_2b_1|$.



The parallelogram can be regarded as a kind of geometrical product of its sides:





The bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$ have the same magnitude but opposite senses of rotation. This can be expressed simply by writing

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

Using the multiplication table of the Clifford algebra $\mathcal{C}\ell_2$ we notice that the Clifford product

$$(a_1\mathbf{e}_1 + a_2\mathbf{e}_2)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\mathbf{e}_{12}$$

of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ is a sum of a scalar $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$ and a bivector $\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12}$. [11] In an equation,

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}. \tag{a}$$

The commutative rule $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ together with the anticommutative rule $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$ implies a relation between $\mathbf{ab}$ and $\mathbf{ba}$. Thus,

$$\mathbf{ba} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b}. \tag{b}$$

Adding and subtracting equations (a) and (b), we find

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) \quad \text{and} \quad \mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab}-\mathbf{ba}).$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a}\parallel\mathbf{b}$, when they commute, $\mathbf{ab} = \mathbf{ba}$, that is, $\mathbf{a}\wedge\mathbf{b} = 0$ or $a_1b_2 = a_2b_1$, and orthogonal, $\mathbf{a}\perp\mathbf{b}$, when they anticommute, $\mathbf{ab} = -\mathbf{ba}$, that is, $\mathbf{a}\cdot\mathbf{b} = 0$. Thus,

$$\begin{aligned}
\mathbf{ab} = \mathbf{ba} \quad &\Longleftrightarrow \quad \mathbf{a}\parallel\mathbf{b} \quad \Longleftrightarrow \quad \mathbf{a}\wedge\mathbf{b} = 0 \quad \Longleftrightarrow \quad \mathbf{ab} = \mathbf{a}\cdot\mathbf{b},\\
\mathbf{ab} = -\mathbf{ba} \quad &\Longleftrightarrow \quad \mathbf{a}\perp\mathbf{b} \quad \Longleftrightarrow \quad \mathbf{a}\cdot\mathbf{b} = 0 \quad \Longleftrightarrow \quad \mathbf{ab} = \mathbf{a}\wedge\mathbf{b}.
\end{aligned}$$

1.11 Components of a vector in given directions

Consider decomposing a vector $\mathbf{r}$ into two components, one parallel to $\mathbf{a}$ and the other parallel to $\mathbf{b}$, where $\mathbf{a}\nparallel\mathbf{b}$. This means determining the coefficients α and β in the decomposition $\mathbf{r} = \alpha\mathbf{a}+\beta\mathbf{b}$. The coefficient α may be obtained by forming the exterior product $\mathbf{r}\wedge\mathbf{b} = (\alpha\mathbf{a}+\beta\mathbf{b})\wedge\mathbf{b}$ and using $\mathbf{b}\wedge\mathbf{b} = 0$; this results in $\mathbf{r}\wedge\mathbf{b} = \alpha(\mathbf{a}\wedge\mathbf{b})$. Similarly, $\mathbf{a}\wedge\mathbf{r} = \beta(\mathbf{a}\wedge\mathbf{b})$. In the last two equations both sides are multiples of $\mathbf{e}_{12}$ and we may write, symbolically, [12]

$$\alpha = \frac{\mathbf{r}\wedge\mathbf{b}}{\mathbf{a}\wedge\mathbf{b}}, \qquad \beta = \frac{\mathbf{a}\wedge\mathbf{r}}{\mathbf{a}\wedge\mathbf{b}}.$$

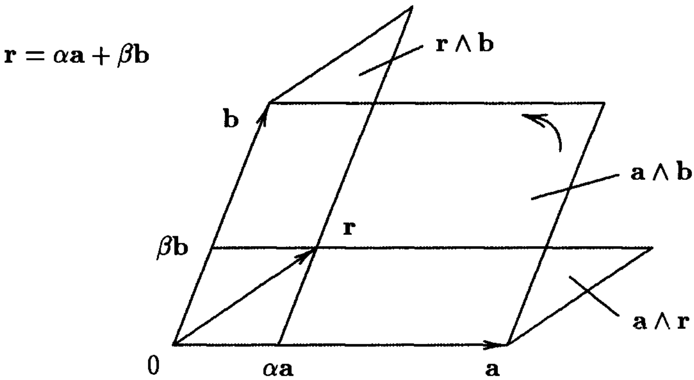
11 The bivector valued exterior product $\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12}$, which represents a plane area, should not be confused with the vector valued cross product $\mathbf{a}\times\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_3$, which represents a line segment.

12 As an element of the exterior algebra $\bigwedge\mathbb{R}^2$ the bivector $\mathbf{a}\wedge\mathbf{b}$ is not invertible. As an element of the Clifford algebra $\mathcal{C}\ell_2$ a non-zero bivector $\mathbf{a}\wedge\mathbf{b}$ is invertible, but since the multiplication in $\mathcal{C}\ell_2$ is non-commutative, it is more appropriate to write

$$\alpha = (\mathbf{r}\wedge\mathbf{b})(\mathbf{a}\wedge\mathbf{b})^{-1} \quad \text{and} \quad \beta = (\mathbf{a}\wedge\mathbf{r})(\mathbf{a}\wedge\mathbf{b})^{-1}.$$

However, since $\mathbf{r}\wedge\mathbf{b}$, $\mathbf{a}\wedge\mathbf{r}$ and $\mathbf{a}\wedge\mathbf{b}$ commute, our notation is also acceptable.

The coefficients α and β could be obtained visually by comparing the oriented areas (instead of lengths) in the following figure:



Exercise 5

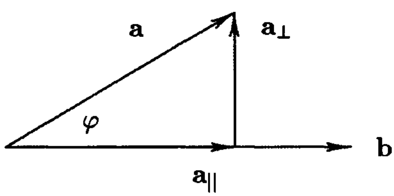
1.12 Perpendicular projections and reflections

Let us calculate the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ when the two vectors diverge by an angle φ, $0 < \varphi < 180°$. The parallel component $\mathbf{a}_{\|}$ is a scalar multiple of the unit vector $\mathbf{b}/|\mathbf{b}|$:

$$\mathbf{a}_{\|} = |\mathbf{a}| \cos\varphi \frac{\mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}||\mathbf{b}| \cos\varphi \frac{\mathbf{b}}{|\mathbf{b}|^2}.$$

In other words, the parallel component $\mathbf{a}_{\|}$ is the scalar product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\varphi$ multiplied by the vector $\mathbf{b}^{-1} = \mathbf{b}/|\mathbf{b}|^2$, called the inverse [13] of the vector $\mathbf{b}$. Thus,

$$\mathbf{a}_{\|} = (\mathbf{a} \cdot \mathbf{b}) \frac{\mathbf{b}}{|\mathbf{b}|^2} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1}.$$



The last formula tells us that the length of $\mathbf{b}$ is irrelevant when projecting into the direction of $\mathbf{b}$.

The perpendicular component $\mathbf{a}_{\perp}$ is given by the difference

$$\begin{aligned}\mathbf{a}_{\perp} &= \mathbf{a} - \mathbf{a}_{\|} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} \\ &= (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} = (\mathbf{a} \wedge \mathbf{b})\mathbf{b}^{-1}.\end{aligned}$$

13 The inverse $\mathbf{b}^{-1}$ of a non-zero vector $\mathbf{b} \in \mathbb{R}^2 \subset C\ell_2$ satisfies $\mathbf{b}^{-1}\mathbf{b} = \mathbf{b}\mathbf{b}^{-1} = 1$ in the Clifford algebra $C\ell_2$. A vector and its inverse are parallel vectors.

Note that the bivector $\mathbf{e}_{12}$ anticommutes with all the vectors in the $\mathbf{e}_1\mathbf{e}_2$-plane, therefore

$$(\mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1} = -\mathbf{b}^{-1}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{b}^{-1}(\mathbf{b}\wedge\mathbf{a}) = -(\mathbf{b}\wedge\mathbf{a})\mathbf{b}^{-1}.$$

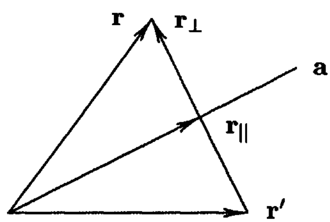
The area of the parallelogram with sides $\mathbf{a}, \mathbf{b}$ is seen to be

$$|\mathbf{a}_\perp\mathbf{b}| = |\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$

where $0 < \varphi < 180°$.

The reflection of $\mathbf{r}$ across the line $\mathbf{a}$ is obtained by sending $\mathbf{r} = \mathbf{r}_\| + \mathbf{r}_\perp$ to $\mathbf{r}' = \mathbf{r}_\| - \mathbf{r}_\perp$, where $\mathbf{r}_\| = (\mathbf{r}\cdot\mathbf{a})\mathbf{a}^{-1}$. The mirror image $\mathbf{r}'$ of $\mathbf{r}$ with respect to $\mathbf{a}$ is then

$$\begin{aligned}\mathbf{r}' &= (\mathbf{r}\cdot\mathbf{a})\mathbf{a}^{-1} - (\mathbf{r}\wedge\mathbf{a})\mathbf{a}^{-1}\\ &= (\mathbf{r}\cdot\mathbf{a} - \mathbf{r}\wedge\mathbf{a})\mathbf{a}^{-1}\\ &= (\mathbf{a}\cdot\mathbf{r} + \mathbf{a}\wedge\mathbf{r})\mathbf{a}^{-1}\\ &= \mathbf{ara}^{-1}\end{aligned}$$



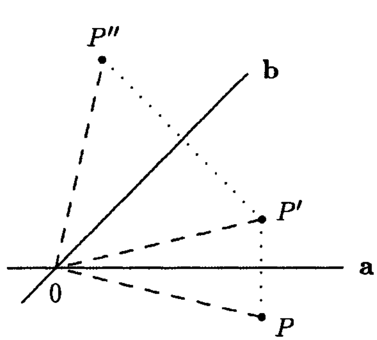
and further

$$\begin{aligned}\mathbf{r}' &= (2\mathbf{a}\cdot\mathbf{r} - \mathbf{ra})\mathbf{a}^{-1}\\ &= 2\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}^2}\mathbf{a} - \mathbf{r}.\end{aligned}$$

The formula $\mathbf{r}' = \mathbf{ara}^{-1}$ can be obtained directly using only commutation properties of the Clifford product: decompose $\mathbf{r} = \mathbf{r}_\| + \mathbf{r}_\perp$, where $\mathbf{ar}_\|\mathbf{a}^{-1} = \mathbf{r}_\|\mathbf{aa}^{-1} = \mathbf{r}_\|$, while $\mathbf{ar}_\perp\mathbf{a}^{-1} = -\mathbf{r}_\perp\mathbf{aa}^{-1} = -\mathbf{r}_\perp$.

The composition of two reflections, first across $\mathbf{a}$ and then across $\mathbf{b}$, is given by

$$\mathbf{r} \to \mathbf{r}' = \mathbf{ara}^{-1} \to \mathbf{r}'' = \mathbf{br}'\mathbf{b}^{-1} = \mathbf{b}(\mathbf{ara}^{-1})\mathbf{b}^{-1} = (\mathbf{ba})\mathbf{r}(\mathbf{ba})^{-1}.$$



The composite of these two reflections is a rotation by twice the angle between $\mathbf{a}$ and $\mathbf{b}$. As a consequence, if a triangle ABC with angles α, β, γ is turned about its vertices A, B, C by the angles $2\alpha, 2\beta, 2\gamma$ in the same direction, the result is an identity rotation.

Exercises 6,7

1.13 Matrix representation of $C\ell_2$

In this chapter we have introduced the Clifford algebra $C\ell_2$ of the Euclidean plane $\mathbb{R}^2$. The Clifford algebra $C\ell_2$ is a 4-dimensional algebra over the reals $\mathbb{R}$. It is isomorphic, as an associative algebra, to the matrix algebra of real 2×2-matrices $\mathrm{Mat}(2, \mathbb{R})$, as can be seen by the correspondences

$$1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{e}_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{e}_{12} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

However, in the Clifford algebra $C\ell_2$ there is more structure than in the matrix algebra $\mathrm{Mat}(2, \mathbb{R})$. In the Clifford algebra $C\ell_2$ we have singled out by definition a privileged subspace, namely the subspace of vectors or 1-vectors $\mathbb{R}^2 \subset C\ell_2$. No similar privileged subspace is incorporated in the definition of the matrix algebra $\mathrm{Mat}(2, \mathbb{R})$. [14]

For arbitrary elements the above correspondences mean that

$$u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12} \simeq \begin{pmatrix} u_0 + u_1 & u_2 + u_{12} \\ u_2 - u_{12} & u_0 - u_1 \end{pmatrix}$$

and

$$\frac{1}{2}[(a+d) + (a-d)\mathbf{e}_1 + (b+c)\mathbf{e}_2 + (b-c)\mathbf{e}_{12}] \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this representation the transpose of a matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\top} = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

corresponds to the *reverse*

$$\tilde{u} = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 - u_{12}\mathbf{e}_{12}$$

14 For instance, we might choose $\mathbf{u}_1 = \sqrt{2}\mathbf{e}_1 + \mathbf{e}_{12}$, $\mathbf{u}_2 = \mathbf{e}_2$. This also results in the commutation relations $\mathbf{u}_1^2 = 1$, $\mathbf{u}_2^2 = 1$, $\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2\mathbf{u}_1 = 0$, which define a different representation of $C\ell_2$ as $\mathrm{Mat}(2, \mathbb{R})$.

of $u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$ in $C\ell_2$. The complementary (or adjoint) matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \left[= (ad-bc)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \quad \text{for} \quad ad-bc \neq 0 \right]$$

corresponds to the *Clifford-conjugate* [15]

$$\bar{u} = u_0 - u_1\mathbf{e}_1 - u_2\mathbf{e}_2 - u_{12}\mathbf{e}_{12}.$$

The reversion and Clifford-conjugation are anti-involutions, that is, involutory anti-automorphisms,

$$\tilde{\tilde{u}} = u, \quad \widetilde{uv} = \tilde{v}\tilde{u},$$
$$\bar{\bar{u}} = u, \quad \overline{uv} = \bar{v}\bar{u}.$$

We still need the *grade involute*

$$\hat{u} = u_0 - u_1\mathbf{e}_1 - u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$$

for which $\hat{u} = \tilde{u}^{-} = \bar{u}^{\sim}$.

Exercises

1. Let $a = \mathbf{e}_2 - \mathbf{e}_{12}$, $b = \mathbf{e}_1 + \mathbf{e}_2$, $c = 1 + \mathbf{e}_2$. Compute ab, ac. What did you learn by completing this computation?
2. Let $a = \mathbf{e}_2 + \mathbf{e}_{12}$, $b = \frac{1}{2}(1 + \mathbf{e}_1)$. Compute ab, ba. What did you learn?
3. Let $a = 1 + \mathbf{e}_1$, $b = -1 + \mathbf{e}_1$, $c = \mathbf{e}_1 + \mathbf{e}_2$. Compute ab, ba, ac, ca, bc and cb. What did you learn?
4. Let $a = \frac{1}{2}(1 + \mathbf{e}_1)$, $b = \mathbf{e}_1 + \mathbf{e}_{12}$. Compute a^2, b^2.
5. Let $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{r} = 5\mathbf{e}_1 - \mathbf{e}_2$. Compute α, β in the decomposition $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$.
6. Let $\mathbf{a} = 8\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2$. Compute $\mathbf{a}_{\|}$, $\mathbf{a}_{\perp}$.
7. Let $\mathbf{r} = 4\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{a} = 3\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2$. Reflect first $\mathbf{r}$ across $\mathbf{a}$ and then the result across $\mathbf{b}$.
8. Show that for any $u \in C\ell_2$, $u\bar{u} = \bar{u}u \in \mathbb{R}$, and that u is invertible, if $u\bar{u} \neq 0$, with inverse
$$u^{-1} = \frac{\bar{u}}{u\bar{u}}.$$
9. Let $u = 1 + \mathbf{e}_1 + \mathbf{e}_{12}$. Compute u^{-1}. Show that $u^{-1} = \hat{u}(u\hat{u})^{-1} \neq (u\hat{u})^{-1}\hat{u}$, $u^{-1} = (\hat{u}u)^{-1}\hat{u} \neq \hat{u}(\hat{u}u)^{-1}$ and $u^{-1} = \tilde{u}(u\tilde{u})^{-1} \neq (u\tilde{u})^{-1}\tilde{u}$, $u^{-1} = (\tilde{u}u)^{-1}\tilde{u} \neq \tilde{u}(\tilde{u}u)^{-1}$.

15 In some countries a vector $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \in \mathbb{R}^2$ is denoted by $\bar{u}$ in handwriting, but this practice clashes with our notation for the Clifford-conjugate.

10. Consider the four anti-involutions of $\mathrm{Mat}(2,\mathbb{R})$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Define two anti-automorphisms α, β to be similar, if there is an intertwining automorphism γ such that $\alpha\gamma = \gamma\beta$. Determine which ones of these four anti-involutions are similar or dissimilar to each other. Hint: keep track of what happens to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with squares I, I, and $-I$.

Remark. In completing the exercises, note that an arbitrary element of $C\ell_2$ is most easily perceived when written in the order of increasing indices as $u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$. ∎

Solutions

1. $ab = ac = 1 - \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{12}$; one can learn that $ab = ac \not\Rightarrow b = c$.
2. $ab = 0$, $ba = \mathbf{e}_2 + \mathbf{e}_{12}$; one can learn that $ab = 0 \not\Rightarrow ba = 0$ (and also that $ba = a \not\Rightarrow b = 1$).
3. $ab = ba = 0$, $ac = 1 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12}$, $ca = 1 + \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{12}$, $bc = 1 - \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_{12}$, $cb = 1 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_{12}$; one can learn that $ab = ba = 0 \not\Rightarrow ac = 0$ or $ca = 0$.
4. $a^2 = a$, $b^2 = 0$.
5. $\mathbf{r} = 2\mathbf{a} + 3\mathbf{b}$.
6. $\mathbf{a}_{\|} = 6\mathbf{e}_1 + 3\mathbf{e}_2$, $\mathbf{a}_{\perp} = 2\mathbf{e}_1 - 4\mathbf{e}_2$.
7. $\mathbf{r}' = \mathbf{a}\mathbf{r}\mathbf{a}^{-1} = 5\mathbf{e}_1$, $\mathbf{r}'' = \mathbf{b}\mathbf{r}'\mathbf{b}^{-1} = 3\mathbf{e}_1 + 4\mathbf{e}_2$.
8. $u\bar{u} = \bar{u}u = u_0^2 - u_1^2 - u_2^2 + u_{12}^2 \in \mathbb{R}$.
9. $u^{-1} = 1 - \mathbf{e}_1 - \mathbf{e}_{12}$ and $(u\hat{u})^{-1}\hat{u} = \tilde{u}(\tilde{u}u)^{-1} = 1 + 3\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_{12}$ and $\hat{u}(\hat{u}u)^{-1} = (u\tilde{u})^{-1}\tilde{u} = 1 + 3\mathbf{e}_1 + 4\mathbf{e}_2 - 5\mathbf{e}_{12}$.
10. Only two of the anti-involutions are similar,

$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix},$$

as can be seen by choosing the intertwining automorphism

$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

for which $\alpha\gamma = \gamma\beta$.

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2

Complex Numbers

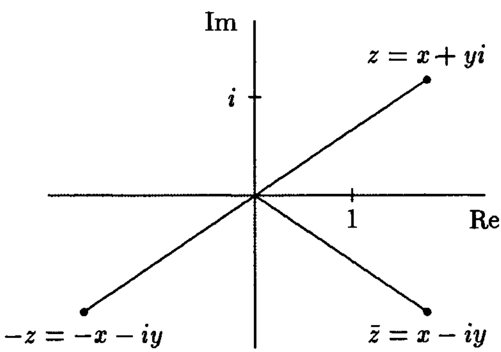
The feature distinguishing the complex numbers from the real numbers is that the complex numbers contain a square root of -1 called the *imaginary unit* $i = \sqrt{-1}$. [1] Complex numbers are of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and i satisfies $i^2 = -1$. The real numbers x, y are called the *real part* $x = \text{Re}(z)$ and the *imaginary part* $y = \text{Im}(z)$. To each ordered pair of real numbers x, y there corresponds a unique complex number $x + iy$.

A complex number $x + iy$ can be represented graphically as a point with rectangular coordinates (x, y). The xy-plane, where the complex numbers are represented, is called the *complex plane* $\mathbb{C}$. Its x-axis is the *real axis* and y-axis the *imaginary axis*.

A complex number $z = x + iy$ has an opposite $-z = -x - iy$ and a *complex conjugate* $\bar{z} = x - iy$, [2] obtained by changing the sign of the imaginary part.



1 Electrical engineers denote the square root of -1 by $j = \sqrt{-1}$.
2 In quantum mechanics the complex conjugate is denoted by $z^* = x - iy$.

The sum of two complex numbers is computed by adding separately the real parts and the imaginary parts:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Addition of complex numbers can be illustrated by the parallelogram law of vector addition.

The product of two complex numbers is usually defined to be

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2),$$

although this result is also a consequence of distributivity, associativity and the replacement $i^2 = -1$.

Examples. 1. $i^3 = -i$, $i^4 = 1$, $i^5 = i$. 2. $(1+i)^2 = 2i$. ∎

The product of a complex number $z = x + iy$ and its complex conjugate $\bar{z} = x - iy$ is a real number $z\bar{z} = x^2 + y^2$. Since this real number is non-zero for $z \neq 0$, we may introduce the inverse

$$z^{-1} = \frac{\bar{z}}{z\bar{z}}$$

or in coordinate form

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

Division is carried out as multiplication by the inverse: $z_1/z_2 = z_1z_2^{-1}$.

If we introduce polar coordinates r, φ in the complex plane by setting $x = r\cos\varphi$ and $y = r\sin\varphi$, then the complex number $z = x + iy$ can be written as

$$z = r(\cos\varphi + i\sin\varphi).$$

This is the *polar form* of z. [3] The distance $r = \sqrt{x^2+y^2}$ from z to 0 is denoted by $|z|$ and called the *norm* of z. Thus [4]

$$|z| = \sqrt{z\bar{z}}.$$

The real number φ is called the *phase-angle* or *argument* of z [sometimes all the real numbers $\varphi + 2\pi k$, $k \in \mathbb{Z}$, are assigned to the same phase-angle].

The familiar addition rules for the sine and cosine result in the polar form of multiplication,

$$z_1z_2 = r_1r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

3 Electrical engineers denote the polar form by $r\underline{/\varphi}$.

4 The scalar product $\mathrm{Re}(z_1\bar{z}_2)$ is compatible with the norm $|z|$. Incidentally, $\mathrm{Im}(z_1\bar{z}_2)$ measures the signed area of the parallelogram determined by z_1 and z_2.

of complex numbers

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1) \quad \text{and} \quad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2).$$

Thus, the norm of a product is the product of the norms,

$$|z_1z_2| = |z_1||z_2|,$$

and the phase-angle of a product is the sum of the phase-angles (mod 2π).

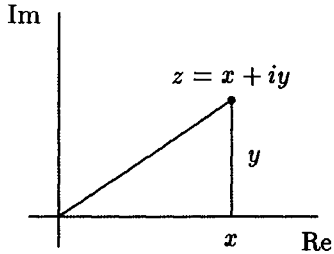
The exponential function can be defined everywhere in the complex plane by

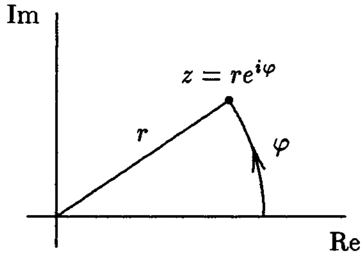
$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots + \frac{z^k}{k!} + \ldots$$

We write $e^z = \exp(z)$. The series expansions of trigonometric functions result in *Euler's formula*

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

which allows us to abbreviate $z = r(\cos\varphi + i\sin\varphi)$ as $z = re^{i\varphi}$.





The exponential form of multiplication seems natural:

$$(r_1e^{i\varphi_1})(r_2e^{i\varphi_2}) = (r_1r_2)e^{i(\varphi_1+\varphi_2)}.$$

Powers and roots are computed as

$$(re^{i\varphi})^n = r^ne^{in\varphi} \quad \text{and} \quad \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r}e^{i\varphi/n+i2\pi k/n}, \ k \in \mathbb{Z}_n.$$

Examples. $(1+i)^{-1} = \frac{1}{2}(1-i)$, $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$, $e^{i\pi/2} = i$. ∎

2.1 The field $\mathbb{C}$ versus the real algebra $\mathbb{C}$

Numbers are elements of a mathematical object called a field. In a field numbers can be both added and multiplied. The usual rules of addition

$a + b = b + a$	commutativity
$(a+b)+c = a+(b+c)$	associativity
$a + 0 = a$	zero 0
$a + (-a) = 0$	opposite $-a$ of a

are satisfied for all numbers a, b, c in a field $\mathbb{F}$. The multiplication satisfies

$$\left.\begin{array}{l}(a+b)c = ac+bc \\ a(b+c) = ab+ac\end{array}\right\} \quad \text{distributivity}$$

$$\begin{array}{ll} (ab)c = a(bc) & \text{associativity} \\ 1a = a & \text{unity } 1 \\ aa^{-1} = 1 & \text{inverse } a^{-1} \text{ of } a \neq 0 \\ ab = ba & \text{commutativity} \end{array}$$

for all numbers a, b, c in a field $\mathbb{F}$. The above rules of addition and multiplication make up the *axioms* of a **field** $\mathbb{F}$.

Examples of fields are the fields of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, rationals $\mathbb{Q}$, and the finite fields $\mathbb{F}_q$ where $q = p^m$ with a prime p. [5]

It is tempting to regard $\mathbb{R}$ as a unique subfield in $\mathbb{C}$. However, $\mathbb{C}$ contains several, infinitely many, subfields isomorphic to $\mathbb{R}$; choosing one means introducing a real linear structure on $\mathbb{C}$, obtained by restricting a in the product $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $(a, b) \to ab$ to be real, $a \in \mathbb{R}$. Such extra structure turns the field $\mathbb{C}$ into a real algebra $\mathbb{C}$.

Definition. An *algebra* over a field $\mathbb{F}$ is a linear space A over $\mathbb{F}$ together with a bilinear [6] function $A \times A \to A$, $(a, b) \to ab$. [7] ∎

To distinguish the field $\mathbb{C}$ from a real algebra $\mathbb{C}$ let us construct $\mathbb{C}$ as the set $\mathbb{R} \times \mathbb{R}$ of all ordered pairs of real numbers $z = (x, y)$ with addition and multiplication defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{and}$$
$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

The set $\mathbb{R} \times \mathbb{R}$ together with the above addition and multiplication rules makes up the *field* $\mathbb{C}$. The imaginary unit $(0, 1)$ satisfies $(0, 1)^2 = (-1, 0)$.

Since $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1x_2, 0)$, the real field $\mathbb{R}$ is contained in $\mathbb{C}$ as a subfield by $\mathbb{R} \to \mathbb{C}$, $x \to (x, 0)$. If we restrict multiplication so that one factor is in this distinguished copy of $\mathbb{R}$,

$$(\lambda, 0)(x, y) = (\lambda x, \lambda y),$$

then we actually introduce a *real linear structure* on the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This

5 The finite fields $\mathbb{F}_q$, where $q = p^m$ with a prime p, are called Galois fields $GF(p^m)$.
6 Bilinear means linear with respect to both arguments. This implies distributivity. In other words, distributivity has no independent meaning for an algebra.
7 Note that associativity is not assumed.

real linear structure allows us to view the field of complex numbers intuitively as the complex plane $\mathbb{C}$. [8]

The above construction of $\mathbb{C}$ as the real linear space $\mathbb{R}^2$ brings in more structure than just the field structure: it makes $\mathbb{C}$ an *algebra* over $\mathbb{R}$. [9] We often identify $\mathbb{R}$ with the subfield $\{(x,0) \mid x \in \mathbb{R}\}$ of $\mathbb{C}$, and denote the standard basis of $\mathbb{R}^2$ by $1 = (1,0)$, $i = (0,1)$ in $\mathbb{C}$.

A function $\alpha : \mathbb{C} \to \mathbb{C}$ is an *automorphism of the field* $\mathbb{C}$ if it preserves addition and multiplication,

$$\begin{aligned} &\alpha(z_1 + z_2) = \alpha(z_1) + \alpha(z_2), \\ &\alpha(z_1 z_2) = \alpha(z_1)\alpha(z_2), \end{aligned}$$

as well as the unity, $\alpha(1) = 1$. A function $\alpha : \mathbb{C} \to \mathbb{C}$ is an *automorphism of the real algebra* $\mathbb{C}$ if it preserves the real linear structure and multiplication (of complex numbers),

$$\begin{aligned} &\alpha(z_1 + z_2) = \alpha(z_1) + \alpha(z_2), \quad \alpha(\lambda z) = \lambda\alpha(z), \quad \lambda \in \mathbb{R}, \\ &\alpha(z_1 z_2) = \alpha(z_1)\alpha(z_2), \end{aligned}$$

as well as the unity, $\alpha(1) = 1$.

The field $\mathbb{C}$ has an infinity of automorphisms. In contrast, the only automorphisms of the real algebra $\mathbb{C}$ are the identity automorphism and complex conjugation.

Theorem. Complex conjugation is the only field automorphism of $\mathbb{C}$ which is different from the identity but preserves a fixed subfield $\mathbb{R}$.

Proof. First, note that $\alpha(i) = \pm i$ for any field automorphism α of $\mathbb{C}$, since $\alpha(i)^2 = \alpha(i^2) = \alpha(-1) = -1$. If $\alpha : \mathbb{C} \to \mathbb{C}$ is a field automorphism such that $\alpha(\mathbb{R}) \subset \mathbb{R}$, then $\alpha(x) = x$ for all $x \in \mathbb{R}$, because the only automorphism of the real field is the identity. It then follows that, for all $x + iy$ with $x, y \in \mathbb{R}$,

$$\alpha(x + iy) = \alpha(x) + \alpha(i)\alpha(y) = x + \alpha(i)y$$

where $\alpha(i) = \pm i$. The case $\alpha(i) = i$ gives the identity automorphism, and the case $\alpha(i) = -i$ gives complex conjugation. ∎

The other automorphisms of the field $\mathbb{C}$ send a real subfield $\mathbb{R}$ onto an isomorphic copy of $\mathbb{R}$, which is necessarily different from the original subfield $\mathbb{R}$. However, any field automorphism of $\mathbb{C}$ fixes point-wise the rational subfield $\mathbb{Q}$.

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8 The geometric view of complex numbers is connected with the structure of $\mathbb{C}$ as a real algebra, and not so much as a field.

9 In the above construction we introduced a field structure into the real linear space $\mathbb{R}^2$ and arrived at an algebra $\mathbb{C}$ over $\mathbb{R}$, or equivalently at a field $\mathbb{C}$ with a distinguished subfield $\mathbb{R}$.

Example. It is known that there is a field automorphism of $\mathbb{C}$ sending $\sqrt{2}$ to $-\sqrt{2}$ and $\sqrt[4]{2}$ to $i\sqrt[4]{2}$, but no one has been able to construct such an automorphism explicitly since its existence proof calls for the axiom of choice. ∎

If a field automorphism of $\mathbb{C}$ is neither the identity nor a complex conjugation, then it sends some irrational numbers outside $\mathbb{R}$, and permutes an infinity of subfields all isomorphic with $\mathbb{R}$. Related to each real subfield there is a unique complex conjugation across that subfield, and all such automorphisms of finite order are complex conjugations for some real subfield. The image $\alpha(\mathbb{R})$ under such an automorphism α of a distinguished real subfield $\mathbb{R}$ is dense in $\mathbb{C}$ [in the topology of the metric $|z| = \sqrt{z\bar{z}}$ given by the complex conjugation across $\mathbb{R}$]. This can be seen as follows: An automorphism α must satisfy $\alpha(rx) = r\alpha(x)$ when $r \in \mathbb{Q}$. So if there is an irrational $x \in \mathbb{R}$ with $t = \alpha(x) \notin \mathbb{R}$, and necessarily $t \notin \mathbb{Q} + i\mathbb{Q}$, the image $\alpha(\mathbb{R})$ of $\mathbb{R}$ contains all numbers of the form $\alpha(r + sx) = r + st$ with $r, s \in \mathbb{Q}$. This is a dense set in $\mathbb{C}$.

The above discussion indicates that **there is no unique complex conjugation in the field of complex numbers**, and that the field structure of $\mathbb{C}$ does not fix by itself the subfield $\mathbb{R}$ of $\mathbb{C}$. The field injection $\mathbb{R} \to \mathbb{C}$ is an extra piece of structure added on top of the field $\mathbb{C}$. If a privileged real subfield $\mathbb{R}$ is singled out in $\mathbb{C}$, it brings along a real linear structure on $\mathbb{C}$, and a unique complex conjugation across $\mathbb{R}$, which then naturally imports a metric structure to $\mathbb{C}$.

Our main interest in complex numbers in this book is $\mathbb{C}$ as a real algebra, not so much as a field.

2.2 The double-ring $^2\mathbb{R}$ of $\mathbb{R}$

There is more than one interesting bilinear product (or algebra structure) on the linear space $\mathbb{R}^2$. For instance, component-wise multiplication

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

results in the double-ring $^2\mathbb{R}$ of $\mathbb{R}$. The only automorphisms of the real algebra $^2\mathbb{R}$ are the identity and the *swap*

$$^2\mathbb{R} \to {}^2\mathbb{R}, \ (\lambda, \mu) \to \mathrm{swap}(\lambda, \mu) = (\mu, \lambda).$$

The swap acts like the complex conjugation of $\mathbb{C}$, since

$$\mathrm{swap}[a(1,1) + b(1,-1)] = a(1,1) - b(1,-1).$$

The multiplicative unity $1 = (1, 1)$ and the reflected element $j = (1, -1)$ are now related by $j^2 = 1$.

Alternatively and equivalently we may consider pairs of real numbers $(a,b) \in \mathbb{R}^2$ as Study numbers

$$a + jb, \quad j^2 = 1, \ j \neq 1.$$

Study numbers have Study conjugate $(a+jb)^- = a - jb$, Lorentz squared norm $(a+jb)(a-jb) = a^2 - b^2$, and the hyperbolic polar form $a + jb = \rho(\cosh\chi + j\sinh\chi)$ for $a^2 - b^2 \geq 0$. [10] In products Lorentz squared norms are preserved and hyperbolic angles added. Study numbers have the matrix representation

$$a + jb \simeq \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Exercise 4

2.3 Representation by means of real 2×2-matrices

Complex numbers were constructed as ordered pairs of real numbers. Thus we can replace

$$z = x + iy \quad \text{in} \quad \mathbb{C} \quad \text{by} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{in} \quad \mathbb{R}^2,$$

making explicit the real linear structure on $\mathbb{C}$. The product of two complex numbers $c = a + ib$ and z,

$$cz = ax - by + i(bx + ay),$$

can be replaced by / factored as

$$\begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One is thus led to consider representing complex numbers by certain real 2×2-matrices in $\mathrm{Mat}(2,\mathbb{R})$: [11]

$$\mathbb{C} \to \mathrm{Mat}(2,\mathbb{R}), \quad a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

10 The linear space $\mathbb{R}^2$ endowed with an indefinite quadratic form $(a,b) \to a^2 - b^2$ is the hyperbolic quadratic space $\mathbb{R}^{1,1}$. The Clifford algebra of $\mathbb{R}^{1,1}$ is $C\ell_{1,1}$ which has Study numbers as the even subalgebra $C\ell^+_{1,1}$.

11 In this matrix representation, the complex conjugate of a complex number becomes the transpose of the matrix and the (squared) norm becomes the determinant. The norm is preserved under similarity transformations, but 'transposition = complex conjugation' is only preserved under similarities by orthogonal matrices.

The multiplicative unity 1 and the imaginary unit i in $\mathbb{C}$ are represented by the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

However, this is not the only linear representation of $\mathbb{C}$ in $\mathrm{Mat}(2,\mathbb{R})$. A similarity transformation by an invertible matrix U, $\det U \neq 0$, sends the representative of the imaginary unit J to another 'imaginary unit' $J' = UJU^{-1}$ in $\mathrm{Mat}(2,\mathbb{R})$.

Example. Choosing $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we find $J' = \left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right)$, and the matrix representation $x + iy \to \left(\begin{smallmatrix} x+y & -2y \\ y & x-y \end{smallmatrix}\right)$. ∎

GEOMETRIC INTERPRETATION OF $i = \sqrt{-1}$

In the rest of this chapter we shall study introduction of complex numbers by means of the Clifford algebra $\mathcal{C}\ell_2$ of the Euclidean plane $\mathbb{R}^2$. This approach gives the imaginary unit $i = \sqrt{-1}$ various geometrical meanings. We will see that i represents

(i) an oriented plane area in $\mathbb{R}^2$,
(ii) a quarter turn of $\mathbb{R}^2$.

The Euclidean plane $\mathbb{R}^2$ has a quadratic form

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}|^2 = x^2 + y^2.$$

We introduce an associative product of vectors such that

$$\mathbf{r}^2 = |\mathbf{r}|^2 \quad \text{or} \quad (x\mathbf{e}_1 + y\mathbf{e}_2)^2 = x^2 + y^2.$$

Using distributivity this results in the multiplication rules

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \quad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1.$$

The element $\mathbf{e}_1\mathbf{e}_2$ satisfies

$$\boxed{(\mathbf{e}_1\mathbf{e}_2)^2 = -1}$$

and therefore cannot be a scalar or a vector. It is an example of a bivector, the *unit bivector*. Denote it for short by $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$.

2.4 $\mathbb{C}$ as the even Clifford algebra $\mathcal{C}\ell_2^+$

The Clifford algebra $\mathcal{C}\ell_2$ is a 4-dimensional real algebra with a basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$. The basis elements obey the multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

The basis elements span the subspaces consisting of [12]

1	$\mathbb{R}$	scalars
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbb{R}^2$	vectors
$\mathbf{e}_{12}$	$\bigwedge^2 \mathbb{R}^2$	bivectors.

Thus, the Clifford algebra $\mathcal{C}\ell_2$ contains copies of $\mathbb{R}$ and $\mathbb{R}^2$, and it is a direct sum of its subspaces of elements of degrees 0,1,2:

$$\mathcal{C}\ell_2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2.$$

The Clifford algebra is also a direct sum $\mathcal{C}\ell_2 = \mathcal{C}\ell_2^+ \oplus \mathcal{C}\ell_2^-$ of its

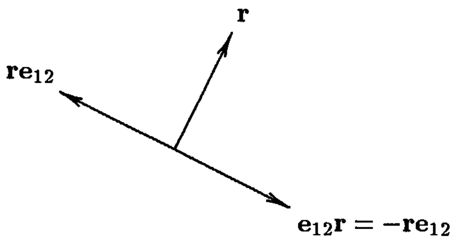
$$\begin{aligned} \text{even part} \quad & \mathcal{C}\ell_2^+ = \mathbb{R} \oplus \textstyle\bigwedge^2 \mathbb{R}^2, \\ \text{odd part} \quad & \mathcal{C}\ell_2^- = \mathbb{R}^2. \end{aligned}$$

The even part is not only a subspace but also a subalgebra. It consists of elements of the form $x + y\mathbf{e}_{12}$ where $x, y \in \mathbb{R}$ and $\mathbf{e}_{12}^2 = -1$. Thus, the even subalgebra $\mathcal{C}\ell_2^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ of $\mathcal{C}\ell_2$ is isomorphic to $\mathbb{C}$. The unit bivector $\mathbf{e}_{12}$ shares the basic property of the square root i of -1, that is $i^2 = -1$, and we could write $i = \mathbf{e}_{12}$. It should be noted, however, that our imaginary unit $\mathbf{e}_{12}$ anticommutes with $\mathbf{e}_1$ and $\mathbf{e}_2$ and thus $\mathbf{e}_{12}$ *anticommutes with every vector in the* $\mathbf{e}_1\mathbf{e}_2$*-plane:* [13]

$$\mathbf{r}\mathbf{e}_{12} = -\mathbf{e}_{12}\mathbf{r} \quad \text{for} \quad \mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \quad \text{and} \quad \mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2.$$

12 In higher dimensions the Clifford algebra $\mathcal{C}\ell_n$ of $\mathbb{R}^n$ is a sum of its subspaces of k-vectors: $\mathcal{C}\ell_n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \ldots \oplus \bigwedge^n \mathbb{R}^n$.

13 In a complex linear space, or complex algebra, where scalars are complex numbers, the imaginary unit commutes with all the vectors, $i\mathbf{r} = \mathbf{r}i$.



2.5 Imaginary unit = the unit bivector

Multiplying the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ by the unit bivector $\mathbf{e}_{12}$ gives another vector $\mathbf{r}\mathbf{e}_{12} = x\mathbf{e}_2 - y\mathbf{e}_1$ which is perpendicular to $\mathbf{r}$. The function $\mathbf{r} \to \mathbf{r}\mathbf{e}_{12}$ is a left turn, and the effect of two left turns $[\mathbf{e}_{12} \cdot \mathbf{e}_{12}]$ is to reverse direction $[-1]$; or, in a more picturesque way, is a U-turn. The statement '$\mathbf{e}_{12}^2 = -1$' is just an arithmetic version of the obvious geometric fact that the sum of two right angles, $90° + 90°$, is a straight angle, $180°$. In the vector plane $\mathbb{R}^2$ the sense of rotation depends on what side the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ is multiplied by $\mathbf{e}_{12}$ so that the rotation $\mathbf{r} \to \mathbf{e}_{12}\mathbf{r} = y\mathbf{e}_1 - x\mathbf{e}_2$ is clockwise and $\mathbf{r} \to \mathbf{r}\mathbf{e}_{12} = -y\mathbf{e}_1 + x\mathbf{e}_2$ is counter-clockwise.

In the complex plane $\mathbb{C} = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ both the rotations sending $z = x + y\mathbf{e}_{12}$ to $\mathbf{e}_{12}z$ and $z\mathbf{e}_{12}$ are counter-clockwise. Multiplying a complex number $z = x + y\mathbf{e}_{12}$ by the unit bivector $\mathbf{e}_{12}$ results in a left turn, $z\mathbf{e}_{12} = -y + x\mathbf{e}_{12}$, and the effect of two left turns $[\mathbf{e}_{12} \cdot \mathbf{e}_{12}]$ is direction reversal $[-1]$; that is a half-turn in the complex plane $\mathbb{C}$:

$$\begin{tikzcd} -z = z\mathbf{e}_{12}^2 & z\mathbf{e}_{12} \arrow[l] \\ {} \arrow[r] & z \arrow[u] \end{tikzcd}$$

The square root of -1 has two distinct geometric roles in $\mathbb{R}^2$: it is the generator of rotations, $i = \mathbf{e}_1\mathbf{e}_2 \in \mathcal{C}\ell_2^+$, and it represents a unit oriented plane area $\mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^2$. [14]

A complex number $z = x + y\mathbf{e}_{12} \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ is a sum of

- a real number $x = \mathrm{Re}(z)$ and
- a bivector $y\mathbf{e}_{12} = \mathbf{e}_{12}\,\mathrm{Im}(z)$.

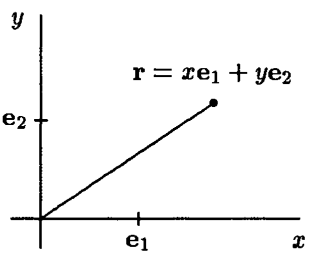
14 In an n-dimensional vector space $\mathbb{R}^n$ rotations can be represented by multiplications in Clifford algebras $\mathcal{C}\ell_n$, while certain simple elements of the exterior algebra $\bigwedge \mathbb{R}^n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \cdots \oplus \bigwedge^n \mathbb{R}^n$ represent oriented subspaces of dimensions $0, 1, 2, \ldots, n$.

2.6 Even and odd parts

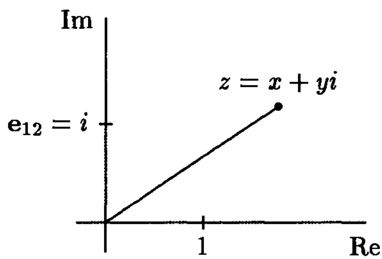
The Clifford algebra $C\ell_2$ of $\mathbb{R}^2$ contains both the complex plane $\mathbb{C}$ and the vector plane $\mathbb{R}^2$ so that

$\mathbb{R}^2$ is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$,
$\mathbb{C}$ is spanned by 1 and $\mathbf{e}_{12}$.

The only common point of the two planes is the zero 0. The two planes are both parts of the same algebra $C\ell_2$. The vector plane $\mathbb{R}^2$ and the complex field $\mathbb{C}$ are incorporated as separate substructures in the Clifford algebra $C\ell_2 = C\ell_2^+ \oplus C\ell_2^-$ so that the complex plane $\mathbb{C}$ is the *even part* $C\ell_2^+$ and the vector plane $\mathbb{R}^2$ is the *odd part* $C\ell_2^-$.



Vector plane $\mathbb{R}^2 = C\ell_2^-$



Complex plane $\mathbb{C} = C\ell_2^+$

The names even and odd mean that the elements are products of an even or odd number of vectors. Parity considerations show that

- complex number times complex number is a complex number,
- vector times complex number is a vector,
- complex number times vector is a vector, and
- vector times vector is a complex number.

The above observations can be expressed by the inclusions

$$C\ell_2^+ C\ell_2^+ \subset C\ell_2^+,$$
$$C\ell_2^- C\ell_2^+ \subset C\ell_2^-,$$
$$C\ell_2^+ C\ell_2^- \subset C\ell_2^-,$$
$$C\ell_2^- C\ell_2^- \subset C\ell_2^+.$$

By writing $(C\ell_2)_0 = C\ell_2^+$ and $(C\ell_2)_1 = C\ell_2^-$, this can be further condensed to $(C\ell_2)_j(C\ell_2)_k \subset (C\ell_2)_{j+k}$, where j, k are added modulo 2. These observations are expressed by saying that the Clifford algebra $C\ell_2$ has an *even-odd grading* or that it is graded over $\mathbb{Z}_2 = \{0, 1\}$. [15]

15 We have already met a $\mathbb{Z}_2$-graded algebra, namely the real algebra $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ with even part $\mathbb{R} = \mathrm{Re}(\mathbb{C})$ and odd part $i\mathbb{R} = i\,\mathrm{Im}(\mathbb{C})$.

2.7 Involutions and the norm

The Clifford algebra $\mathcal{C}\ell_2$ has three involutions similar to complex conjugation in $\mathbb{C}$. For an element $u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 \in \mathcal{C}\ell_2$, $\langle u\rangle_k \in \bigwedge^k \mathbb{R}^2$, we define

$$\begin{array}{ll} \text{grade involution} & \hat{u} = \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2, \\ \text{reversion} & \tilde{u} = \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2, \\ \text{Clifford-conjugation} & \bar{u} = \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2. \end{array}$$

The grade involution is an automorphism, $\widehat{uv} = \hat{u}\hat{v}$, while the reversion and the Clifford-conjugation are anti-automorphisms, $\widetilde{uv} = \tilde{v}\tilde{u}$, $\overline{uv} = \bar{v}\bar{u}$.

For a complex number $z = x + y\mathbf{e}_{12}$ the complex conjugation $z \to \bar{z} = x - y\mathbf{e}_{12}$ is a restriction of the Clifford-conjugation $u \to \bar{u}$ in $\mathcal{C}\ell_2$ and also of the reversion $u \to \tilde{u}$ in $\mathcal{C}\ell_2$. Likewise, the norm $|z| = \sqrt{x^2 + y^2}$ in $\mathbb{C}$, obtained as the square root of $z\bar{z} = x^2 + y^2$, is a restriction of the norm $|u| = \sqrt{\langle u\tilde{u}\rangle_0}$ in $\mathcal{C}\ell_2$.

A complex number is a product of its norm $r = |z|$ and its phase-factor $\cos\varphi + \mathbf{e}_{12}\sin\varphi$, where $x = r\cos\varphi$ and $y = r\sin\varphi$. The expression $z = r(\cos\varphi + \mathbf{e}_{12}\sin\varphi)$ can be abbreviated as $z = r\exp(\mathbf{e}_{12}\varphi)$, and read as '$r$ in phase φ.'

2.8 Vectors multiplied by complex numbers

The product of a vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ and a unit complex number $e^{\mathbf{i}\varphi} = \cos\varphi + \mathbf{i}\sin\varphi$, where for short $\mathbf{i} = \mathbf{e}_{12}$, is another vector in the $\mathbf{e}_1\mathbf{e}_2$-plane:

$$\mathbf{r}\cos\varphi + \mathbf{r}\mathbf{i}\sin\varphi = \mathbf{r}e^{\mathbf{i}\varphi}.$$

The vector $\mathbf{r}\mathbf{i} = x\mathbf{e}_2 - y\mathbf{e}_1$ is perpendicular to $\mathbf{r}$ so that a rotation to the left by $\pi/2$ carries $\mathbf{r}$ to $\mathbf{r}\mathbf{i}$.

Since the unit bivector $\mathbf{i}$ anticommutes with every vector $\mathbf{r}$ in the $\mathbf{e}_1\mathbf{e}_2$-plane, the rotated vector could also be expressed as

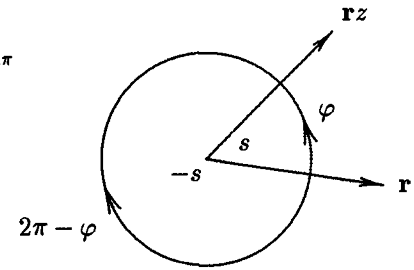
$$\mathbf{r}\cos\varphi + \mathbf{r}\mathbf{i}\sin\varphi = \mathbf{r}\cos\varphi - \mathbf{i}\mathbf{r}\sin\varphi = e^{-\mathbf{i}\varphi}\mathbf{r}.$$

Furthermore, we have $\cos\varphi + \mathbf{i}\sin\varphi = (\cos\frac{\varphi}{2} + \mathbf{i}\sin\frac{\varphi}{2})^2$ and thus the rotated vector also has the form $s^{-1}\mathbf{r}s$ where $s = e^{\mathbf{i}\varphi/2}$ and $s^{-1} = e^{-\mathbf{i}\varphi/2}$. The rotation of $\mathbf{r}$ to the left by the angle φ will then result in $\mathbf{r}z = z^{-1}\mathbf{r} = s^{-1}\mathbf{r}s$ where $z = e^{\mathbf{i}\varphi}$, $z^{-1} = e^{-\mathbf{i}\varphi}$ and $s^2 = z$. There are two complex numbers s and $-s$ which result in the same rotation $s^{-1}\mathbf{r}s = (-s)^{-1}\mathbf{r}(-s)$. In other words, there are two complex numbers which produce the same final result but via different actions.

$s = e^{i\varphi/2}$

$-s = e^{-i(2\pi-\varphi)/2} = e^{i\varphi/2}e^{-i\pi}$

$\boxed{e^{i\pi} = -1}$



2.9 The group Spin(2)

The unit complex numbers $z \in \mathbb{C}$, $|z| = 1$, form the *unit circle* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, which with multiplication of complex numbers as the product becomes the *unitary group* $U(1) = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$. A counter-clockwise rotation of the complex plane $\mathbb{C}$ by an angle φ can be represented by complex number multiplication:

$$x + iy \to (\cos\varphi + i\sin\varphi)(x + iy), \quad \cos\varphi + i\sin\varphi \in U(1).$$

A counter-clockwise rotation of the vector plane $\mathbb{R}^2$ by an angle φ can be represented by a matrix multiplication:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$$

where $SO(2) = \{R \in \mathrm{Mat}(2,\mathbb{R}) \mid R^{\mathsf{T}}R = I,\ \det R = 1\}$, the rotation group. The rotation group $SO(2)$ is isomorphic to the unitary group $U(1)$.

Rotations of $\mathbb{R}^2$ can also be represented by Clifford multiplication: [16]

$$x\mathbf{e}_1 + y\mathbf{e}_2 \to (\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})^{-1}(x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})$$

where $\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2} \in \mathbf{Spin}(2) = \{s \in C\ell_2^+ \mid s\bar{s} = 1\}$, the spin group. The fact that two opposite elements of the spin group **Spin**(2) represent the same rotation in $SO(2)$ is expressed by saying that **Spin**(2) is a *two-fold* [17] cover of $SO(2)$, and written as $\mathbf{Spin}(2)/\{\pm 1\} \simeq SO(2)$. Although $SO(2)$ and **Spin**(2) act differently on $\mathbb{R}^2$, they are isomorphic as abstract groups, that is,

16 We use this particular form to represent the rotation because the expression $x\mathbf{e}_1 + y\mathbf{e}_2 \to (\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})^{-1}(x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})$ can be generalized to higher dimensions. The expression $x\mathbf{e}_1 + y\mathbf{e}_2 \to (x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\varphi + \mathbf{e}_{12}\sin\varphi)$ is not generalizable to higher-dimensional rotations.

17 You are already familiar with two-fold covers: 1. A position of the hands of your watch corresponds to two positions of the Sun. 2. A rotating mirror turns half the angle of the image. 3. Circulating a coin one full turn around another makes the coin turn twice around its center.

$\mathbf{Spin}(2) \simeq SO(2)$. [18]

Exercise 6

History

Imaginary numbers first appeared around 1540, when Tartaglia and Cardano expressed real roots of a cubic equation in terms of conjugate complex numbers. The first one to represent complex numbers by points on a plane was a Norwegian surveyor, Caspar Wessel, in 1798. He posited an imaginary axis perpendicular to the axis of real numbers. This configuration came to be known as the Argand diagram, although Argand's contribution was an interpretation of $i = \sqrt{-1}$ as a rotation by a right angle in the plane. Complex numbers got their name from Gauss, and their formal definition as pairs of real numbers is due to Hamilton in 1833 (first published 1837).

Exercises

1. $(3+4i)^{-1}$, $\sqrt{3+4i}$, $\sqrt[4]{-4}$, $\sqrt[3]{-i}$, $\log(-1+i)$.
2. Let $z_k = e^{i\,2\pi k/n}$, $k = 1, 2, \ldots, n-1$. Compute $(1-z_1)(1-z_2)\cdots(1-z_{n-1})$.
3. An *ordering* of a field $\mathbb{F}$ is an assignment of a subset $P \subset \mathbb{F}$ such that

   (i) $0 \notin P$,
   (ii) for all non-zero $a \in \mathbb{F}$ either $a \in P$ or $-a \in P$, but not both,
   (iii) $a + b \in P$ and $ab \in P$ for all $a, b \in P$.

   It is customary to call P the set of *positive* numbers, and the set $-P = \{-a \mid a \in P\}$ the set of *negative* numbers. The statement $a - b \in P$ is also written $a > b$ (and $a - b \in P \cup \{0\}$ is written $a \geq b$). Show that the field $\mathbb{C}$ cannot be ordered.
4. Two automorphisms α, β of an algebra are similar if there exists an intertwining automorphism γ such that $\alpha\gamma = \gamma\beta$. The identity automorphism is similar only to itself.

a) Show that the two involutions of the real algebra $\mathbb{C}$ are dissimilar, and that the two involutions of the real algebra ${}^2\mathbb{R}$ are dissimilar.

b) Show that the two involutions $\alpha(\lambda, \mu) = (\mu, \lambda)$ and $\beta(\lambda, \mu) = (\bar{\mu}, \bar{\lambda})$ are similar involutions of the real or complex algebra ${}^2\mathbb{C}$ [that is, find an intertwining automorphism γ of ${}^2\mathbb{C}$ such that $\alpha\gamma = \gamma\beta$].

5. A rotation is called rational if it sends a vector with rational coordinates to

18 Both $SO(2)$ and $\mathbf{Spin}(2)$ are homeomorphic to S^1.

another vector with rational coordinates. Determine all the rational rotations of $\mathbb{R}^2$. Hint: $R \in SO(2) \setminus \{-I\}$ can be written in the form $R = (I + A)(I - A)^{-1}$ where $A^\top = -A$.

6. Write $\tilde{u} = \langle u \rangle_0 + \langle u \rangle_1 - \langle u \rangle_2$ for $u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 \in C\ell_2$, $\langle u \rangle_k \in \bigwedge^k \mathbb{R}^2$. Let $\mathbf{Pin}(2) = \{u \in C\ell_2 \mid \tilde{u}u = 1\}$, $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \to R(\mathbf{x}) = u\mathbf{x}u^{-1}$, and $O(2) = \{R \in \mathrm{Mat}(2,\mathbb{R}) \mid R^\top R = I\}$. Show that $\mathbf{Pin}(2)/\{\pm 1\} \simeq O(2)$ and $\mathbf{Pin}(2) \simeq O(2)$.
7. Show that a 2-dimensional real algebra with unity 1 is both commutative and associative. Hint: First show that there is a basis $\{1, a\}$ such that $a^2 = \alpha 1,\ \alpha \in \mathbb{R}$.
8. Show that a 2-dimensional real algebra with unity 1 and no zero-divisors [$ab = 0$ implies $a = 0$ or $b = 0$] is isomorphic to $\mathbb{C}$.

Solutions

1. $\frac{1}{5}(3 - 4i),\ \pm(2 + i),\ \pm 1 \pm i,\ \sqrt[3]{-i} = \{i, \pm\frac{\sqrt{3}}{2} - i\frac{1}{2}\}$, $\log(-1 + i) = \frac{1}{2}\log 2 + i\frac{3\pi}{4} + i\,2\pi k$.
2. Note that the roots of $x^n - 1 = 0$ are $z_k = e^{i\,2\pi k/n}$, $k = 0, 1, \ldots, n-1$. Therefore $(x - z_0)(x - z_1)(x - z_2)\cdots(x - z_{n-1}) = x^n - 1$. Define $f(x) = (x - z_1)(x - z_2)\cdots(x - z_{n-1})$ which equals
$$f(x) = \frac{x^n - 1}{x - 1} \quad \text{for} \quad x \neq 1$$
and $f(x) = x^{n-1} + \ldots + x + 1$ in general. Compute $f(1) = n$.
3. In an ordered field non-zero numbers have positive squares, and the sum of such squares is positive, and therefore non-zero. The equality $i^2 + 1 = 0$ in $\mathbb{C}$ can also be written as $i^2 + 1^2 = 0$, which excludes the inequality $i^2 + 1^2 > 0$. Consequently, it is impossible to order the field $\mathbb{C}$.
4. b) Choose $\gamma(\lambda, \mu) = (\bar{\lambda}, \mu)$ or $\gamma(\lambda, \mu) = (\lambda, \bar{\mu})$ to find $\alpha\gamma = \gamma\beta$.

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3

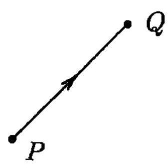
Bivectors and the Exterior Algebra

There are other kinds of directed quantities besides vectors, most notably bivectors. For instance, a moment of a force, angular velocity of a rotating body, and magnetic induction can be described with bivectors. In three dimensions bivectors are dual to vectors, and their use can be circumvented. Scalars, vectors, bivectors and the volume element span the exterior algebra $\bigwedge \mathbb{R}^3$, which provides a multivector structure for the Clifford algebra $C\ell_3$ of the Euclidean space $\mathbb{R}^3$.

3.1 Bivectors as directed plane segments

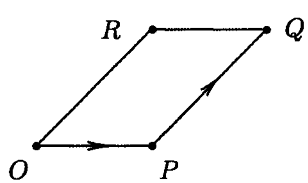
In three dimensions bivectors are oriented plane segments, which have a direction and a magnitude, the area of the plane segment. Two bivectors have the same direction if they are on parallel planes (the same attitude) and are similarly oriented (the same sense of rotation).

Vector (directed line segment)



1. magnitude (length of PQ)
2. direction
   - attitude (line PQ)
   - orientation (toward the point Q)

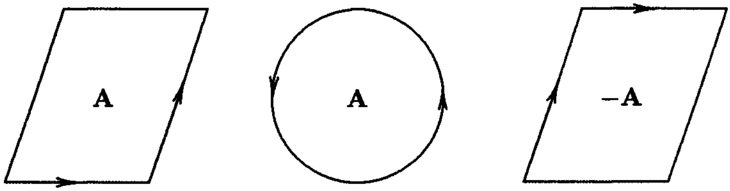
Bivector (directed plane segment)



1. magnitude (area of $OPQR$)
2. direction
   - attitude (plane OPQ)
   - orientation (sense of rotation)

Bivectors are denoted by boldface capital letters $\mathbf{A}, \mathbf{B}$, etc. [1] The area or norm of a bivector $\mathbf{A}$ is denoted by $|\mathbf{A}|$. Two bivectors $\mathbf{A}$ and $\mathbf{B}$ in parallel planes have the same attitude, and we write $\mathbf{A} \parallel \mathbf{B}$. Parallel bivectors $\mathbf{A}$ and $\mathbf{B}$ can be regarded as directed angles turning either the same way, $\mathbf{A} \uparrow\uparrow \mathbf{B}$, or the opposite way, $\mathbf{A} \uparrow\downarrow \mathbf{B}$. If two plane segments have the same area and the same direction (= parallel planes with the same sense of rotation), then the bivectors are equal:

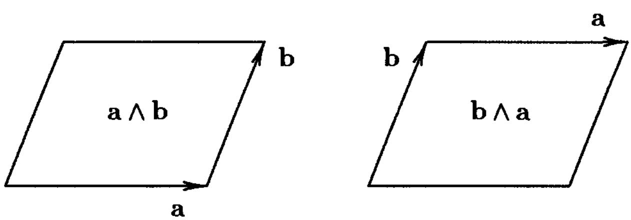
$$\mathbf{A} = \mathbf{B} \iff |\mathbf{A}| = |\mathbf{B}| \quad \text{and} \quad \mathbf{A} \uparrow\uparrow \mathbf{B}$$



A bivector $\mathbf{A}$ and its *opposite* $-\mathbf{A}$ are of equal area and parallel, but have opposite orientations. A *unit bivector* $\mathbf{A}$ has area one, $|\mathbf{A}| = 1$.

The shape of the area is irrelevant.

Representing a bivector as an oriented parallelogram suggests that a bivector can be thought of as a geometrical product of vectors along its sides. With this in mind we introduce the *exterior product* $\mathbf{a} \wedge \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ as the bivector obtained by sweeping $\mathbf{b}$ along $\mathbf{a}$.



The bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$ have the same area and the same attitude but opposite senses of rotations. This can be simply expressed by writing

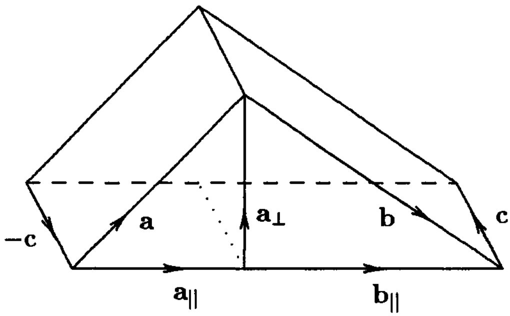
$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

3.2 Addition of bivectors

The geometric interpretation of bivector addition is most easily seen when the bivectors are expressed in terms of the exterior product with a common

1 In handwriting, bivectors can be distinguished by an angle on top of the letter, A, B.

vector factor. In three dimensions this is always possible because any two planes will either be parallel or intersect along a common line. [2] Thus let $\mathbf{A} = \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{B} = \mathbf{b} \wedge \mathbf{c}$; then the bivector $\mathbf{A} + \mathbf{B}$ is defined so that $\mathbf{A} + \mathbf{B} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} + \mathbf{b}) \wedge \mathbf{c}$. The geometric significance of this can be depicted as follows:



By decomposing the vectors $\mathbf{a}$ and $\mathbf{b}$ into components parallel and perpendicular to $\mathbf{a} + \mathbf{b}$, [3] so that

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} \quad \text{and} \quad \mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$$

where $\mathbf{b}_{\perp} = -\mathbf{a}_{\perp}$, we are able to reduce the general addition of bivectors in three dimensions to the addition of coplanar bivectors. This is evident in the equality

$$\mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a}_{||} + \mathbf{b}_{||}) \wedge \mathbf{c} = \mathbf{a}_{||} \wedge \mathbf{c} + \mathbf{b}_{||} \wedge \mathbf{c}.$$

3.3 Basis of the linear space of bivectors

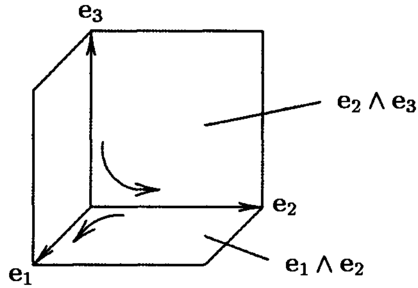
Bivectors can be added and multiplied by scalars. This way the set of bivectors becomes a linear space, denoted by $\bigwedge^2 \mathbb{R}^3$. A basis for the linear space $\bigwedge^2 \mathbb{R}^3$ can be constructed by means of a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the linear space $\mathbb{R}^3$. The oriented plane segments of the coordinate planes, obtained by taking the exterior products

$$\mathbf{e}_1 \wedge \mathbf{e}_2, \ \mathbf{e}_1 \wedge \mathbf{e}_3, \ \mathbf{e}_2 \wedge \mathbf{e}_3,$$

2 The two bivectors are first translated in the affine space $\mathbb{R}^3$ so that they induce opposite orientations to their common edge, that is, the terminal side of $\mathbf{A} = \mathbf{a} \wedge \mathbf{c}$ is opposite to the initial side of $\mathbf{B} = (-\mathbf{c}) \wedge \mathbf{b}$.

3 A depiction of addition of bivectors does not require a metric, or perpendicular components. It is sufficient that one component of both $\mathbf{a}$ and $\mathbf{b}$ is parallel to $\mathbf{a} + \mathbf{b}$, so that the two components sum up to $\mathbf{a} + \mathbf{b}$, while the other component can be any non-parallel component.

form a basis for the linear space of bivectors $\bigwedge^2 \mathbb{R}^3$.



An arbitrary bivector is a linear combination of the basis elements,

$$\mathbf{B} = B_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + B_{13}\mathbf{e}_1 \wedge \mathbf{e}_3 + B_{23}\mathbf{e}_2 \wedge \mathbf{e}_3,$$

and such linear combinations form the space of bivectors $\bigwedge^2 \mathbb{R}^3$. [4] The construction of bivectors calls only for a linear structure, and no metric is needed.

The scalar product on a Euclidean space $\mathbb{R}^3$ extends to a symmetric bilinear product on the space of bivectors $\bigwedge^2 \mathbb{R}^3$,

$$<\mathbf{x}_1 \wedge \mathbf{x}_2, \mathbf{y}_1 \wedge \mathbf{y}_2> = \begin{vmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \mathbf{x}_1 \cdot \mathbf{y}_2 \\ \mathbf{x}_2 \cdot \mathbf{y}_1 & \mathbf{x}_2 \cdot \mathbf{y}_2 \end{vmatrix}.$$

In particular, $<\mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{b}> = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. The norm or area of $\mathbf{B} = B_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + B_{13}\mathbf{e}_1 \wedge \mathbf{e}_3 + B_{23}\mathbf{e}_2 \wedge \mathbf{e}_3$ is seen to be

$$|\mathbf{B}| = \sqrt{<\mathbf{B}, \mathbf{B}>} = \sqrt{B_{12}^2 + B_{13}^2 + B_{23}^2}.$$

3.4 The oriented volume element

The exterior product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ of three vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ and $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ represents the oriented volume of the parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

It is an element of the 1-dimensional linear space of 3-vectors $\bigwedge^3 \mathbb{R}^3$ with basis $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. The exterior product is associative,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}),$$

4 In three dimensions all bivectors are simple, that is, they are exterior products of two vectors, $\mathbf{B} = \mathbf{x} \wedge \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. This is no longer true in four dimensions; for instance $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$ is not simple.

and antisymmetric,

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} &= \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{a} = \mathbf{c} \wedge \mathbf{a} \wedge \mathbf{b} \\ &= -\mathbf{c} \wedge \mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{c} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \wedge \mathbf{c}\end{aligned}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

The exterior product of the orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ is the unit oriented volume element $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \in \bigwedge^3 \mathbb{R}^3$. The norm or volume $|\mathbf{V}|$ of a 3-vector [5]

$$\mathbf{V} = V\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

is $|\mathbf{V}| = |V|$, that is, $|V\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3| = V$ for $V \geq 0$ and $|V\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3| = -V$ for $V < 0$.

More formally, the scalar product on $\mathbb{R}^3$ extends to a symmetric bilinear product on $\bigwedge^3 \mathbb{R}^3$ by

$$<\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3, \mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \mathbf{y}_3> = \begin{vmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \mathbf{x}_1 \cdot \mathbf{y}_2 & \mathbf{x}_1 \cdot \mathbf{y}_3 \\ \mathbf{x}_2 \cdot \mathbf{y}_1 & \mathbf{x}_2 \cdot \mathbf{y}_2 & \mathbf{x}_2 \cdot \mathbf{y}_3 \\ \mathbf{x}_3 \cdot \mathbf{y}_1 & \mathbf{x}_3 \cdot \mathbf{y}_2 & \mathbf{x}_3 \cdot \mathbf{y}_3 \end{vmatrix}$$

giving the norm as $|\mathbf{V}| = \sqrt{<\mathbf{V}, \mathbf{V}>}$.

3.5 The cross product

Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. The bivector

$$\mathbf{a} \wedge \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3b_1 - a_1b_3)\mathbf{e}_3 \wedge \mathbf{e}_1 + (a_1b_2 - a_2b_1)\mathbf{e}_1 \wedge \mathbf{e}_2$$

can be expressed as a 'determinant'

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} \mathbf{e}_2 \wedge \mathbf{e}_3 & \mathbf{e}_3 \wedge \mathbf{e}_1 & \mathbf{e}_1 \wedge \mathbf{e}_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

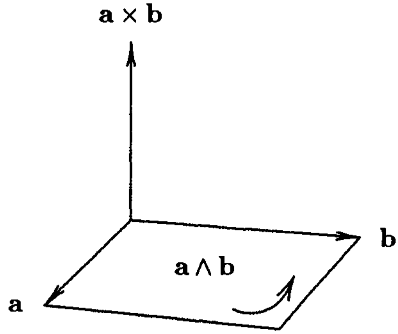
It is customary to introduce a vector with the same coordinates. Thus, we define the *cross product*

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

of $\mathbf{a}$ and $\mathbf{b}$. The cross product can also be represented by a 'determinant'

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

5 V is a real number, positive or negative, while $\mathbf{V}$ is a 3-vector. The usual volume is $|V|$.



The direction of $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane of $\mathbf{a}\wedge\mathbf{b}$ and the length/norm of $\mathbf{a}\times\mathbf{b}$ equals the area/norm of $\mathbf{a}\wedge\mathbf{b}$,

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$

where φ, $0 \le \varphi \le 180°$, is the angle between $\mathbf{a}$ and $\mathbf{b}$.

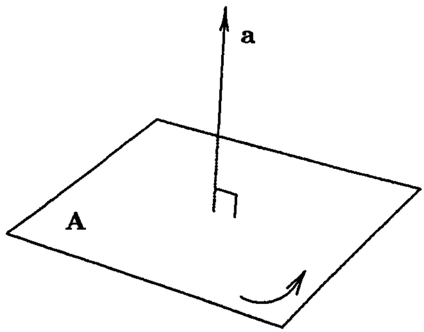
In spite of the resemblance between the determinant expressions for the exterior product $\mathbf{a}\wedge\mathbf{b}$ and the cross product $\mathbf{a}\times\mathbf{b}$ there is a difference: the exterior product does not require a metric while the cross product requires or induces a metric. The metric gets involved in positioning the vector $\mathbf{a}\times\mathbf{b}$ perpendicular to the bivector $\mathbf{a}\wedge\mathbf{b}$.

3.6 The Hodge dual

Since the vector space $\mathbb{R}^3$ and the bivector space $\bigwedge^2\mathbb{R}^3$ are both of dimension 3, they are linearly isomorphic. We can use the metric on the vector space $\mathbb{R}^3$ to set up a standard isomorphism between the two linear spaces, the Hodge dual sending a vector $\mathbf{a}\in\mathbb{R}^3$ to a bivector $\star\mathbf{a}\in\bigwedge^2\mathbb{R}^3$, defined by

$$\mathbf{b}\wedge\star\mathbf{a} = (\mathbf{b}\cdot\mathbf{a})\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3 \quad \text{for all} \quad \mathbf{b}\in\mathbb{R}^3.$$

The Hodge dual depends not only on the metric but also on the choice of orientation – it is customary to use a right-handed and orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.



Vector $\mathbf{a}$ and its dual bivector $\mathbf{A} = \mathbf{a}\mathbf{e}_{123}$

Thus, we have assigned to each vector

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{R}^3$$

a bivector

$$\mathbf{A} = \star\mathbf{a} = a_1\mathbf{e}_2 \wedge \mathbf{e}_3 + a_2\mathbf{e}_3 \wedge \mathbf{e}_1 + a_3\mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^3.$$

Using the induced metric on the bivector space $\bigwedge^2 \mathbb{R}^3$ we can extend the Hodge dual to a mapping sending a bivector $\mathbf{A} \in \bigwedge^2 \mathbb{R}^3$ to a vector $\star\mathbf{A} \in \mathbb{R}^3$, defined by

$$\mathbf{B} \wedge \star\mathbf{A} = \langle\mathbf{B}, \mathbf{A}\rangle \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \quad \text{for all} \quad \mathbf{B} \in \bigwedge^2 \mathbb{R}^3.$$

Using duality, the relation between the cross product and the exterior product can be written as [6]

$$\mathbf{a} \wedge \mathbf{b} = \star(\mathbf{a} \times \mathbf{b}),$$
$$\mathbf{a} \times \mathbf{b} = \star(\mathbf{a} \wedge \mathbf{b}).$$

6 In terms of the Clifford algebra $C\ell_3$ the relation between the exterior product and the cross product can be written as

$$\mathbf{a} \wedge \mathbf{b} = (\mathbf{a} \times \mathbf{b})\mathbf{e}_{123},$$
$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})\mathbf{e}_{123}.$$

The metric gets involved in multiplying by $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Using the Clifford algebra $C\ell_3$ the Hodge dual can be computed as $\star u = \tilde{u}\mathbf{e}_{123}$. This gives rise to the *Clifford dual* defined as $u\mathbf{e}_{123}$ for $u \in C\ell_3$. Later we will see that in actual computations the Clifford dual is more convenient than the Hodge dual (although in three dimensions the Hodge dual happens to be symmetric/involutory).

3.7 The exterior algebra and the Clifford algebra

The exterior algebra $\bigwedge \mathbb{R}^3$ of the linear space $\mathbb{R}^3$ is a direct sum of the

subspaces of		with basis
scalars	$\mathbb{R}$	1
vectors	$\mathbb{R}^3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
bivectors	$\bigwedge^2 \mathbb{R}^3$	$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3$
volume elements	$\bigwedge^3 \mathbb{R}^3$	$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$

We also write $\mathbb{R} = \bigwedge^0 \mathbb{R}^3$ and $\mathbb{R}^3 = \bigwedge^1 \mathbb{R}^3$. Thus, $\bigwedge \mathbb{R}^3$ is a direct sum of its subspaces of homogeneous degrees $0, 1, 2, 3$:

$$\bigwedge \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \overset{2}{\bigwedge} \mathbb{R}^3 \oplus \overset{3}{\bigwedge} \mathbb{R}^3.$$

The dimensions of $\mathbb{R}, \mathbb{R}^3$, $\bigwedge^2 \mathbb{R}^3$, $\bigwedge^3 \mathbb{R}^3$ and $\bigwedge \mathbb{R}^3$ are $1, 3, 3, 1$ and $2^3 = 8$, respectively.

The exterior algebra $\bigwedge \mathbb{R}^3$ is an associative algebra with unity 1 satisfying

$$\boxed{\begin{array}{l} \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i \quad \text{for} \quad i \neq j \\ \mathbf{e}_i \wedge \mathbf{e}_i = 0 \end{array}}$$

for a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the linear space $\mathbb{R}^3$. The exterior product of two homogeneous elements satisfies

$$\mathbf{a} \wedge \mathbf{b} \in \overset{i+j}{\bigwedge} \mathbb{R}^3 \quad \text{for} \quad \mathbf{a} \in \overset{i}{\bigwedge} \mathbb{R}^3,\ \mathbf{b} \in \overset{j}{\bigwedge} \mathbb{R}^3.$$

The product of two elements u and v in the Clifford algebra $\mathcal{C}\ell_3$ of the Euclidean space $\mathbb{R}^3$ is denoted by juxtaposition, uv, to distinguish it from the exterior product $u \wedge v$. An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the Euclidean space $\mathbb{R}^3 \subset \mathcal{C}\ell_3$ satisfies [7]

$$\boxed{\begin{array}{l} \mathbf{e}_i \mathbf{e}_j = -\mathbf{e}_j \mathbf{e}_i \quad \text{for} \quad i \neq j \\ \mathbf{e}_i \mathbf{e}_i = 1 \end{array}}$$

7 These rules were invented by W.K. Clifford in 1882. In an earlier paper Clifford 1878 had considered an associative algebra of dimension 8 with the rules $\mathbf{e}_i \mathbf{e}_i = -1$ for $i = 1, 2, 3$.

and generates a basis of $C\ell_3$, corresponding to a basis of $\bigwedge\mathbb{R}^3$,

$C\ell_3$	$\bigwedge\mathbb{R}^3$
1	1
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
$\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_1\wedge\mathbf{e}_3$, $\mathbf{e}_2\wedge\mathbf{e}_3$
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$

The above correspondences induce an identification of the linear spaces $C\ell_3$ and $\bigwedge\mathbb{R}^3$, and we shall write

$$C\ell_3 = \mathbb{R}\oplus\mathbb{R}^3\oplus\overset{2}{\bigwedge}\mathbb{R}^3\oplus\overset{3}{\bigwedge}\mathbb{R}^3.$$

This decomposition introduces a *multivector structure* into the Clifford algebra $C\ell_3$. The multivector structure is unique, that is, an arbitrary element $u\in C\ell_3$ can be uniquely decomposed into a sum of k-vectors, the k-vector parts $\langle u\rangle_k$ of u,

$$u = \langle u\rangle_0+\langle u\rangle_1+\langle u\rangle_2+\langle u\rangle_3 \quad \text{where} \quad \langle u\rangle_k\in\overset{k}{\bigwedge}\mathbb{R}^3.$$

3.8 The Clifford product of two vectors

A new kind of product called the *Clifford product* of vectors $\mathbf{a}$ and $\mathbf{b}$ is obtained by adding the scalar $\mathbf{a}\cdot\mathbf{b}$ and the bivector $\mathbf{a}\wedge\mathbf{b}$:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}.$$

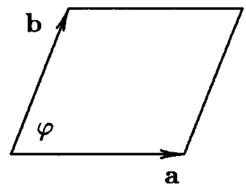
The commutative rule $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ together with the anticommutative rule $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$ implies a relation between $\mathbf{ab}$ and $\mathbf{ba}$. Thus,

$$\mathbf{ba} = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\wedge\mathbf{b}.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a}\parallel\mathbf{b}$, when their product is commutative, $\mathbf{ab}=\mathbf{ba}$, and perpendicular, $\mathbf{a}\perp\mathbf{b}$, when their product is anticommutative, $\mathbf{ab}=-\mathbf{ba}$.

Note that if $\mathbf{a}$ is decomposed into components parallel, $\mathbf{a}_{\|}$, and perpendicular, $\mathbf{a}_{\perp}$, to $\mathbf{b}$, then $\mathbf{ab} = \mathbf{a}_{\|}\mathbf{b}+\mathbf{a}_{\perp}\mathbf{b} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}$.

$$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba})$$
$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba})$$
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\varphi$$
$$|\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$



Compute the product $\mathbf{abba}$ to get $\mathbf{a}^2\mathbf{b}^2 = (\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2$ and use $(\mathbf{a}\wedge\mathbf{b})^2 = -|\mathbf{a}\wedge\mathbf{b}|^2$ to obtain the identity

$$\mathbf{a}^2\mathbf{b}^2 = (\mathbf{a}\cdot\mathbf{b})^2 + |\mathbf{a}\wedge\mathbf{b}|^2.$$

3.9 Even and odd parts

The Clifford algebra is, like the exterior algebra, a direct sum of two of its subspaces,

$$\begin{array}{ll} \text{the even part} & \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3, \\ \text{the odd part} & \mathbb{R}^3\oplus\bigwedge^3\mathbb{R}^3. \end{array}$$

For both algebras the even part is also a subalgebra. The even subalgebra $(\bigwedge\mathbb{R}^3)^+ = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3$ of $\bigwedge\mathbb{R}^3$ is commutative, but the even subalgebra $\mathcal{C}\ell_3^+ = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3$ of $\mathcal{C}\ell_3$ is not commutative; instead it is isomorphic to the quaternion algebra: $\mathbb{H}\simeq\mathcal{C}\ell_3^+$. The odd parts are denoted by $\mathcal{C}\ell_3^-$ and $(\bigwedge\mathbb{R}^3)^-$.

3.10 The center

The center of an algebra consists of those elements which commute with all the elements of the algebra. The center $\mathrm{Cen}(\mathcal{C}\ell_3) = \mathbb{R}\oplus\bigwedge^3\mathbb{R}^3$ of $\mathcal{C}\ell_3$ is isomorphic to $\mathbb{C}$, and the center of $\bigwedge\mathbb{R}^3$ is $\mathrm{Cen}(\bigwedge\mathbb{R}^3) = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3\oplus\bigwedge^3\mathbb{R}^3$.

3.11 Gradings and the multivector structure

The exterior products of homogeneous elements satisfy the relations

$$\mathbf{a}\wedge\mathbf{b}\in\bigwedge^{i+j}\mathbb{R}^3 \quad\text{for}\quad \mathbf{a}\in\bigwedge^{i}\mathbb{R}^3 \quad\text{and}\quad \mathbf{b}\in\bigwedge^{j}\mathbb{R}^3.$$

Such a property of an algebra is usually referred to by saying that the algebra is graded over the index group $\mathbb{Z}$. We shall refer to this property of the exterior algebra $\bigwedge\mathbb{R}^3$ as the *dimension grading*, because simple homogeneous elements represent subspaces of specified dimension. The homogeneous elements in $\bigwedge \mathbb{R}^3$ satisfy

$$\mathbf{a} \wedge \mathbf{b} = (-1)^{ij} \mathbf{b} \wedge \mathbf{a} \quad \text{for} \quad \mathbf{a} \in \overset{i}{\bigwedge} \mathbb{R}^3,\ \mathbf{b} \in \overset{j}{\bigwedge} \mathbb{R}^3,$$

that is, the exterior algebra $\bigwedge \mathbb{R}^3$ is *graded commutative.* [8]

The Clifford products of even and odd subspaces satisfy the inclusion relations

$$\begin{aligned} &\mathcal{C}\ell_3^+ \mathcal{C}\ell_3^+ \subset \mathcal{C}\ell_3^+, \quad &\mathcal{C}\ell_3^+ \mathcal{C}\ell_3^- \subset \mathcal{C}\ell_3^-, \\ &\mathcal{C}\ell_3^- \mathcal{C}\ell_3^+ \subset \mathcal{C}\ell_3^-, \quad &\mathcal{C}\ell_3^- \mathcal{C}\ell_3^- \subset \mathcal{C}\ell_3^+. \end{aligned}$$

These relations can be summarized by saying that the Clifford algebra $\mathcal{C}\ell_3$ has an *even-odd grading*, or that it is graded over the index group $\mathbb{Z}_2 = \{0, 1\}$.

The exterior algebra $\bigwedge \mathbb{R}^3$ is also even-odd graded.

The Clifford algebra $\mathcal{C}\ell_3$ is not graded over $\mathbb{Z}$. However, we can reconstruct the exterior product from the Clifford product in a unique manner. We shall refer to the dimension grading of the associated exterior algebra by saying that the Clifford algebra has a *multivector structure*. Recall that $\mathbb{R}$ and $\mathbb{R}^3$ have, by definition, unique copies in $\mathcal{C}\ell_3$. The exterior product of two vectors equals the antisymmetric part of their Clifford product,

$$\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx}) \in \overset{2}{\bigwedge} \mathbb{R}^3 \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,$$

whence the space of bivectors $\bigwedge^2 \mathbb{R}^3$ has a unique copy in $\mathcal{C}\ell_3$. The subspace of 3-vectors $\bigwedge^3 \mathbb{R}^3$ can be uniquely reconstructed within $\mathcal{C}\ell_3$ by a completely antisymmetrized Clifford product

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{6}(\mathbf{xyz} + \mathbf{yzx} + \mathbf{zxy} - \mathbf{zyx} - \mathbf{xzy} - \mathbf{yxz}) \in \overset{3}{\bigwedge} \mathbb{R}^3$$

of three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$.

Thus, we have established a linear isomorphism sending $\bigwedge \mathbb{R}^3$ to $\mathcal{C}\ell_3$ defined

8 The graded opposite algebra of $\bigwedge \mathbb{R}^3$ is the linear space $\bigwedge \mathbb{R}^3$ with a new product $u \circ v$ defined by

$$(u_0 + u_1) \circ (v_0 + v_1) = v_0 \wedge u_0 + v_0 \wedge u_1 + v_1 \wedge u_0 - v_1 \wedge u_1$$

for $u_0, v_0 \in (\bigwedge \mathbb{R}^3)^+$ and $u_1, v_1 \in (\bigwedge \mathbb{R}^3)^-$. Since $\bigwedge \mathbb{R}^3$ is graded commutative, that is $u \circ v = u \wedge v$, the graded opposite of $\bigwedge \mathbb{R}^3$ is just $\bigwedge \mathbb{R}^3$.

for k-vectors:

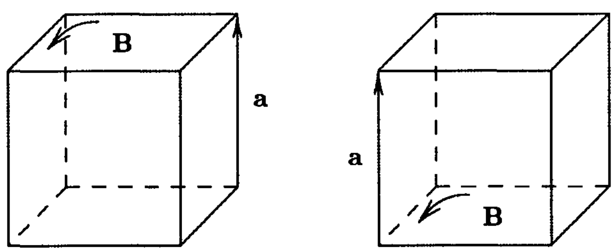
$\bigwedge\mathbb{R}^3$		$C\ell_3$
α	$=$	$\alpha \in \mathbb{R}$
$\mathbf{x}$	$=$	$\mathbf{x} \in \mathbb{R}^3$
$\mathbf{x} \wedge \mathbf{y}$	$=$	$\frac{1}{2}(\mathbf{xy} - \mathbf{yx}) \in \bigwedge^2 \mathbb{R}^3$
$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$	$=$	$\frac{1}{6}(\mathbf{xyz} + \mathbf{yzx} + \mathbf{zxy} - \mathbf{zyx} - \mathbf{xzy} - \mathbf{yxz}) \in \bigwedge^3 \mathbb{R}^3$

There is another construction of the subspace of 3-vectors $\bigwedge^3 \mathbb{R}^3$, obtained by using the reversion, $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{2}(\mathbf{xyz} - \mathbf{zyx}) \in \bigwedge^3 \mathbb{R}^3$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, related to the following recursive construction, via an intermediate step in $\bigwedge^2 \mathbb{R}^3$:

$$\mathbf{x} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{xB} + \mathbf{Bx}) \in \bigwedge^3 \mathbb{R}^3 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3, \ \mathbf{B} \in \bigwedge^2 \mathbb{R}^3.$$

3.12 Products of vectors and bivectors, visualization

A vector $\mathbf{a} \in \mathbb{R}^3$ and a bivector $\mathbf{B} \in \bigwedge \mathbb{R}^3$ can be multiplied to give a 3-vector $\mathbf{a} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{a} \in \bigwedge^3 \mathbb{R}^3$. The exterior product of a vector and a bivector can be depicted as an oriented volume:



$\mathbf{a} \wedge \mathbf{B}$ $\qquad$ $\mathbf{B} \wedge \mathbf{a}$

The orientation is obtained by putting the arrows in succession. The commutativity of the exterior product $\mathbf{a} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{a}$ means that the screws of $\mathbf{a} \wedge \mathbf{B}$ and $\mathbf{B} \wedge \mathbf{a}$ can be rotated onto each other (without reflection).

A vector $\mathbf{x} \in \mathbb{R}^3$ and a bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$ can also be multiplied so that the result is a vector $\mathbf{B} \llcorner \mathbf{x} \in \mathbb{R}^3$. Consider a vector $\mathbf{x}$ tilted by an angle φ out of the plane of a bivector $\mathbf{B}$. Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{x}$ in the plane of $\mathbf{B}$. Then $|\mathbf{a}| = |\mathbf{x}| \cos\varphi$. The right *contraction* of the bivector $\mathbf{B}$ by the vector $\mathbf{x}$ is a vector $\mathbf{y} = \mathbf{B} \llcorner \mathbf{x}$ in the plane of $\mathbf{B}$ such that

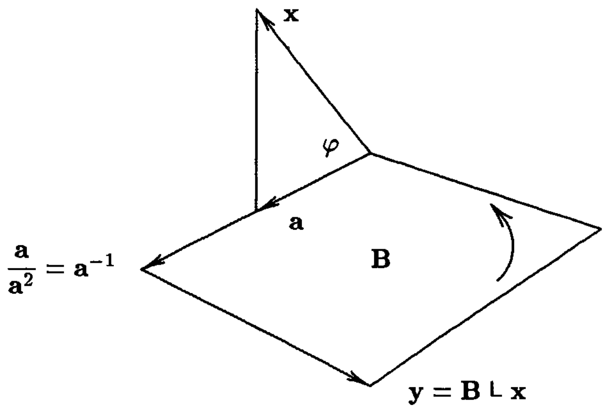
(i) $|\mathbf{y}| = |\mathbf{B}||\mathbf{a}|$,

(ii) $\mathbf{y} \perp \mathbf{a}$ and $\mathbf{a} \wedge \mathbf{y} \uparrow\uparrow \mathbf{B}$.

By convention, we agree that

$$\mathbf{x} \rfloor \mathbf{B} = -\mathbf{B} \llcorner \mathbf{x},$$

that is, the left and right contractions have opposite signs.



[The inverse vector $\mathbf{a}^{-1}$ of $\mathbf{a}$ has a geometrical meaning in this figure: it gives the area of the rectangle, $|\mathbf{B}| = |\mathbf{a}^{-1}||\mathbf{y}|$.]

Write $\mathbf{x}_{\|} = \mathbf{a}$ and $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\|}$. Then $\mathbf{x} \rfloor \mathbf{B} = \mathbf{x}_{\|}\mathbf{B}$ and $\mathbf{x} \wedge \mathbf{B} = \mathbf{x}_{\perp}\mathbf{B}$ so that

$\mathbf{x}_{\|} = (\mathbf{x} \rfloor \mathbf{B})\mathbf{B}^{-1}$	parallel component
$\mathbf{x}_{\perp} = (\mathbf{x} \wedge \mathbf{B})\mathbf{B}^{-1}$	perpendicular component

where $\mathbf{B}^{-1} = \mathbf{B}/\mathbf{B}^2$, $\mathbf{B}^2 = -|\mathbf{B}|^2$.

3.13 Contractions and the derivation

The Clifford product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a sum of a scalar $\mathbf{a} \cdot \mathbf{b}$ and a bivector $\mathbf{a} \wedge \mathbf{b}$,

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b},$$

so that the terms on the right hand side can be recaptured from the Clifford product:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \qquad \mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}).$$

The product of a vector $\mathbf{a}$ and a bivector $\mathbf{B}$ is a sum of a vector and a 3-vector:

$$\mathbf{aB} = \mathbf{a} \rfloor \mathbf{B} + \mathbf{a} \wedge \mathbf{B}$$

where

$$\mathbf{a} \rfloor \mathbf{B} = \frac{1}{2}(\mathbf{aB} - \mathbf{Ba}), \qquad \mathbf{a} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{aB} + \mathbf{Ba}).$$

In general, the Clifford product of a vector $\mathbf{x} \in \mathbb{R}^3$ and an arbitrary element $u \in C\ell_3$ can be decomposed into a sum of the left contraction and the exterior product as follows: [9]

$$\mathbf{x}u = \mathbf{x} \rfloor u + \mathbf{x} \wedge u$$

where we can write, in the case where u is a k-vector in $\bigwedge^k \mathbb{R}^3$,

$$\begin{aligned} \mathbf{x} \rfloor u &= \tfrac{1}{2}(\mathbf{x}u - (-1)^k u\mathbf{x}) \in \textstyle\bigwedge^{k-1} \mathbb{R}^3, \\ \mathbf{x} \wedge u &= \tfrac{1}{2}(\mathbf{x}u + (-1)^k u\mathbf{x}) \in \textstyle\bigwedge^{k+1} \mathbb{R}^3. \end{aligned}$$

The exterior product and the left contraction by a homogeneous element, respectively, raise or lower the degree, that is,

$$\mathbf{a} \wedge \mathbf{b} \in \bigwedge^{i+j} \mathbb{R}^3, \qquad \mathbf{a} \rfloor \mathbf{b} \in \bigwedge^{j-i} \mathbb{R}^3$$

for $\mathbf{a} \in \bigwedge^i \mathbb{R}^3$ and $\mathbf{b} \in \bigwedge^j \mathbb{R}^3$.

The left contraction can be obtained from the exterior product and the Clifford product as follows:

$$u \rfloor v = [u \wedge (v\mathbf{e}_{123})]\mathbf{e}_{123}^{-1}.$$

This means that the left contraction is dual to the exterior product. The *left contraction* can be directly defined by its characteristic properties

1) $\mathbf{x} \rfloor \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$,
2) $\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$,
3) $(u \wedge v) \rfloor w = u \rfloor (v \rfloor w)$,

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $u, v, w \in \bigwedge \mathbb{R}^3$. Recalling that $\hat{u} = (-1)^k u$ for $u \in \bigwedge^k \mathbb{R}^3$, the second rule can also be written as

$$\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + (-1)^k u \wedge (\mathbf{x} \rfloor v),$$

when $u \in \bigwedge^k \mathbb{R}^3$. The second rule means that the left contraction by a vector is a *derivation* of the exterior algebra $\bigwedge \mathbb{R}^3$. It happens that the left contraction by a vector is also a derivation of the Clifford algebra, that is,

$$\mathbf{x} \rfloor (uv) = (\mathbf{x} \rfloor u)v + \hat{u}(\mathbf{x} \rfloor v) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3, \ u, v \in C\ell_3.$$

9 A scalar product on $\mathbb{R}^3 \subset \bigwedge \mathbb{R}^3$ induces a contraction on $\bigwedge \mathbb{R}^3$ which can be used to introduce a new product $\mathbf{x}u = \mathbf{x} \rfloor u + \mathbf{x} \wedge u$ for $\mathbf{x} \in \mathbb{R}^3$ and $u \in \bigwedge \mathbb{R}^3$, which extends by linearity and associativity to all of $\bigwedge \mathbb{R}^3$. The linear space $\bigwedge \mathbb{R}^3$ provided with this new product is the Clifford algebra $C\ell_3$.

3.14 The Clifford algebra versus the exterior algebra

Both the Clifford algebra $\mathcal{C}\ell_3$ and the exterior algebra $\bigwedge \mathbb{R}^3$ contain a copy of $\mathbb{R}^3$, which enables application of calculations to the geometry of $\mathbb{R}^3$. The feature distinguishing $\mathcal{C}\ell_3$ from $\bigwedge \mathbb{R}^3$ is that the Clifford multiplication of vectors preserves the norm, $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, whereas $|\mathbf{a} \wedge \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. The equality $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ enables more calculations to be carried out in $\mathbb{R}^3$, most notably rotations become represented as operations within one algebra, the Clifford algebra $\mathcal{C}\ell_3$.

Historical survey

The exterior algebra $\bigwedge \mathbb{R}^3$ of the linear space $\mathbb{R}^3$ was constructed by Grassmann in 1844. Grassmann's exterior algebra $\bigwedge \mathbb{R}^3$ has a basis

$$\begin{array}{l} 1 \\ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \\ \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3 \\ \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \end{array}$$

satisfying the multiplication rules

$$\begin{array}{ll} \mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i & \text{for} \quad i \neq j, \\ \mathbf{e}_i \wedge \mathbf{e}_i = 0. & \end{array}$$

Clifford introduced a new product into the exterior algebra; he kept the first rule

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j,$$

that is $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$, but replaced the second rule by

$$\begin{array}{lll} \mathbf{e}_i\mathbf{e}_i = 1 & \text{in 1882,} & \text{and} \\ \mathbf{e}_i\mathbf{e}_i = -1 & \text{in 1878.} & \end{array}$$

These two algebras generated are Clifford's geometric algebras

$$\mathcal{C}\ell_3 = \mathcal{C}\ell_{3,0} \simeq \mathrm{Mat}(2, \mathbb{C}) \quad \text{and} \quad \mathcal{C}\ell_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$$

of the positive definite and negative definite quadratic spaces $\mathbb{R}^3 = \mathbb{R}^{3,0}$ and $\mathbb{R}^{0,3}$, respectively.

Exercises

1. Find the area of the triangle with vertices $(1, -4, -6)$, $(5, -4, -2)$ and $(0, 0, 0)$.

2. Find the volume of the parallelepiped with edges $\mathbf{a} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$.
3. Compute the square of the volume element $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ (square with respect to the Clifford product).
4. Show that $\mathbf{e}_{123}$ commutes with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
5. Find the inverse of the bivector $\mathbf{B} = 3\mathbf{e}_{12} + \mathbf{e}_{23}$ (inverse with respect to the Clifford product).
6. Let $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 7\mathbf{e}_3$ and $\mathbf{B} = 4\mathbf{e}_{12} + 5\mathbf{e}_{13} - \mathbf{e}_{23}$. Compute $\mathbf{a} \wedge \mathbf{B}$ and $\mathbf{a} \lrcorner \mathbf{B}$.
7. Let $\mathbf{a} = 3\mathbf{e}_1 + 4\mathbf{e}_2 + 7\mathbf{e}_3$ and $\mathbf{B} = 7\mathbf{e}_{12} + \mathbf{e}_{13}$. Compute the perpendicular and parallel components of $\mathbf{a}$ in the plane of $\mathbf{B}$.
8. Show that the Clifford product of a bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$ and an arbitrary element $u \in \mathcal{C}\ell_3$ can be decomposed as
$$\mathbf{B}u = \mathbf{B} \lrcorner u + \frac{1}{2}(\mathbf{B}u - u\mathbf{B}) + \mathbf{B} \wedge u.$$
9. Reconstruct the dot product $\mathbf{a} \cdot \mathbf{b}$ with the help of the cross product $\mathbf{a} \times \mathbf{b}$ and the exterior product $\mathbf{a} \wedge \mathbf{b}$. Hint: $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - \mathbf{a}^2\mathbf{b}$.

Define the right contraction by $u \llcorner v = \mathbf{e}_{123}^{-1}[(\mathbf{e}_{123}u) \wedge v]$ for $u, v \in \mathcal{C}\ell_3$.

10. Show that the following properties – the characteristic properties – of the right contraction hold:
$$\begin{aligned} &1) \quad \mathbf{x} \llcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}, \\ &2) \quad (u \wedge v) \llcorner \mathbf{x} = u \wedge (v \llcorner \mathbf{x}) + (u \llcorner \mathbf{x}) \wedge \hat{v}, \\ &3) \quad u \llcorner (v \wedge w) = (u \llcorner v) \llcorner w, \end{aligned}$$
for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $u, v, w \in \bigwedge \mathbb{R}^3$.
11. Show that $\mathbf{a} \llcorner \mathbf{b} \in \bigwedge^{i-j} \mathbb{R}^3$ for $\mathbf{a} \in \bigwedge^i \mathbb{R}^3$ and $\mathbf{b} \in \bigwedge^j \mathbb{R}^3$.
12. Show that $(u \lrcorner v) \llcorner w = u \lrcorner (v \llcorner w)$.
13. Show that $u \lrcorner v = \star(\star^{-1}(v) \wedge \tilde{u})$ and $u \llcorner v = \star^{-1}(\tilde{v} \wedge \star(u))$.
14. Show that
$$u\mathbf{x} = u \llcorner \mathbf{x} + u \wedge \mathbf{x}$$
where, for a k-vector $u \in \bigwedge^k \mathbb{R}^3$,
$$\begin{aligned} u \llcorner \mathbf{x} &= \tfrac{1}{2}(u\mathbf{x} - (-1)^k \mathbf{x}u) \in \textstyle\bigwedge^{k-1} \mathbb{R}^3, \\ u \wedge \mathbf{x} &= \tfrac{1}{2}(u\mathbf{x} + (-1)^k \mathbf{x}u) \in \textstyle\bigwedge^{k+1} \mathbb{R}^3. \end{aligned}$$
15. Show that $u \wedge v - v \wedge u \in \bigwedge^2 \mathbb{R}^3$ and $uv - vu \in \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$.

Let $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$, $u = 1 + \mathbf{a} + \mathbf{B}$.

16. The exterior inverse of u is $u^{\wedge(-1)} = 1 - \mathbf{a} - \mathbf{B} + \alpha \mathbf{a} \wedge \mathbf{B}$ with some $\alpha \in \mathbb{R}$. Determine α. Hint: use power series or $u \wedge u^{\wedge(-1)} = 1$.
17. The exterior square root of u is $u^{\wedge(1/2)} = 1 + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{B} + \beta \mathbf{a} \wedge \mathbf{B}$ with some $\beta \in \mathbb{R}$. Determine β. Hint: $u^{\wedge(1/2)} \wedge u^{\wedge(1/2)} = u$.
18. Show that $1 \rfloor u = u$ for all $u \in \bigwedge \mathbb{R}^3$.

Solutions

1. $\mathbf{a} = \mathbf{e}_1 - 4\mathbf{e}_2 - 6\mathbf{e}_3$, $\mathbf{b} = 5\mathbf{e}_1 - 4\mathbf{e}_2 - 2\mathbf{e}_3$, $\mathbf{a} \wedge \mathbf{b} = 16\mathbf{e}_{12} + 28\mathbf{e}_{13} - 16\mathbf{e}_{23}$, $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}| = \frac{1}{2}\sqrt{16^2 + 28^2 + 16^2} = 18$.
2. $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -7\mathbf{e}_{123}$, $|\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}| = 7$.
3. $\mathbf{e}_{123}^2 = -1$.
5. $\mathbf{B}^2 = -10$, $|\mathbf{B}| = \sqrt{10}$, $\mathbf{B}^{-1} = -\frac{1}{10}(3\mathbf{e}_{12} + \mathbf{e}_{23})$.
6. $\mathbf{a} \wedge \mathbf{B} = 11\mathbf{e}_{123}$, $\mathbf{a} \rfloor \mathbf{B} = -47\mathbf{e}_1 + 15\mathbf{e}_2 + 7\mathbf{e}_3$.
7. $\mathbf{a}_\perp = -0.9\mathbf{e}_2 + 6.3\mathbf{e}_3$, $\mathbf{a}_\| = 3\mathbf{e}_1 + 4.9\mathbf{e}_2 + 0.7\mathbf{e}_3$.
9. Take a wedge product with $\mathbf{b}$ to obtain $(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b} = (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \wedge \mathbf{b})$, and

$$\mathbf{a} \cdot \mathbf{b} = \frac{(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} \quad \text{for} \quad \mathbf{a} \nparallel \mathbf{b}$$

(the division is carried out in the Clifford algebra $\mathcal{C}\ell_3$, or it is just a ratio of two parallel bivectors).

16. $\alpha = 2$.
17. $\beta = -\frac{1}{4}$.
18. $1 \rfloor u = (1 \wedge 1) \rfloor u = 1 \rfloor (1 \rfloor u)$ and so the contraction by 1 is a projection with eigenvalues 0 and 1. The only idempotents of $\bigwedge \mathbb{R}^3$ are 0 and 1, and so $1 \rfloor u = 0$ or $1 \rfloor u = u$, identically. The latter must be chosen, since $1 \rfloor (\mathbf{x} \cdot \mathbf{y}) = 1 \rfloor (\mathbf{x} \rfloor \mathbf{y}) = (1 \wedge \mathbf{x}) \rfloor \mathbf{y} = \mathbf{x} \rfloor \mathbf{y} = \mathbf{x} \cdot \mathbf{y} \neq 0$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

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4
Pauli Spin Matrices and Spinors

In classical mechanics kinetic energy $\frac{1}{2}mv^2 = \frac{p^2}{2m}$, $\vec{p} = m\vec{v}$, and potential energy $W = W(\vec{r})$ sum up to the total energy [1]

$$E = \frac{p^2}{2m} + W.$$

Inserting differential operators for total energy and momentum,

$$E = i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} = -i\hbar\nabla,$$

into the above equation results in the *Schrödinger equation* [2]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W\psi,$$

a quantum mechanical description of the electron. The Schrödinger equation explains all atomic phenomena except those involving magnetism and relativity.

The wave function ψ is complex valued, $\psi(\vec{r}, t) \in \mathbb{C}$. The square norm $|\psi|^2$ integrated over a region in space gives the probability of finding the electron in that region. [3]

The Stern & Gerlach experiment, in 1922, showed that a beam of silver atoms splits in two in a magnetic field [there were two distinct spots on the screen, instead of a smear of silver along a line]. Uhlenbeck & Goudsmit in 1925 proposed that silver atoms and the electron have an intrinsic angular momentum, the *spin*. The spin interacts with the magnetic field, and the electron goes up or down according as the spin is parallel or opposite to the vertical magnetic field.

1 This holds in a conservative system.
2 The Schrödinger equation arose out of the hypothesis that if light has both wave and particle properties, then perhaps particles might have wave properties such as interference and diffraction.
3 This is the Born interpretation.

In an electromagnetic field $\vec{E}, \vec{B}$ with potentials $V, \vec{A}$ the Schrödinger equation becomes [4]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[(-i\hbar\nabla - e\vec{A})^2]\psi - eV\psi, \tag{1}$$

or after 'squaring'

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[-\hbar^2\nabla^2 + e^2A^2 + i\hbar e(\nabla\cdot\vec{A} + \vec{A}\cdot\nabla)]\psi - eV\psi.$$

This equation does not yet involve the spin of the electron. The differential operator, known as the generalized momentum,

$$\vec{\pi} = \vec{p} - e\vec{A} \quad \text{where} \quad \vec{p} = -i\hbar\nabla$$

is such that its components $\pi_k = p_k - eA_k$ satisfy the commutation relations

$$\pi_1\pi_2 - \pi_2\pi_1 = i\hbar eB_3 \quad \text{(permute 1, 2, 3 cyclically)}.$$

Pauli 1927 introduced the spin into quantum mechanics by adding a new term into the Schrödinger equation. The *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy

$$\sigma_1\sigma_2 = i\sigma_3 \quad \text{(permute 1, 2, 3 cyclically)}$$

and the anticommutation relations

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I.$$

Applying the above commutation and anticommutation relations, and temporarily using the old-fashioned notation

$$\vec{\sigma}\cdot\vec{\pi} = \sigma_1\pi_1 + \sigma_2\pi_2 + \sigma_3\pi_3,$$

we may see that

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \pi^2 - \hbar e(\vec{\sigma}\cdot\vec{B})$$

where

$$\pi^2 = p^2 + e^2A^2 - e(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}).$$

Pauli replaced π^2 by $(\vec{\sigma}\cdot\vec{\pi})^2$ in equation (1):

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[\pi^2 - \hbar e(\vec{\sigma}\cdot\vec{B})]\psi - eV\psi.$$

4 A Schrödinger equation with $W = 0$ is brought into this form by a gauge transformation $\psi(\vec{r}, t) \to \varphi(\vec{r}, t)e^{i\alpha(\vec{r}, t)}$, when $eV = \hbar\frac{\partial\alpha}{\partial t}$ and $e\vec{A} = \hbar\nabla\alpha$.

This Schrödinger-Pauli equation describes the spin by virtue of the term

$$\frac{\hbar e}{2m}(\vec{\sigma}\cdot\vec{B}).$$

The matrix $\vec{\sigma}\cdot\vec{B}$ operates on two-component column matrices with entries in $\mathbb{C}$. The wave function sends space-time points to *Pauli spinors*

$$\psi(\vec{r},t)=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad \psi_1,\psi_2\in\mathbb{C},$$

that is, it has values in the complex linear space $\mathbb{C}^2$.

The Schrödinger-Pauli equation in the Clifford algebra $C\ell_3$. The multiplication rules of the Pauli spin matrices $\sigma_1,\sigma_2,\sigma_3\in\mathrm{Mat}(2,\mathbb{C})$ imply the matrix identity

$$(\vec{\sigma}\cdot\vec{B})^2=(B_1^2+B_2^2+B_3^2)I.$$

Thus, we may regard the set of traceless Hermitian matrices as a Euclidean space $\mathbb{R}^3$ with an orthonormal basis $\{\sigma_1,\sigma_2,\sigma_3\}$.

The length (of the representative) of a vector $\vec{B}$ is preserved under a similarity transformation $U(\vec{\sigma}\cdot\vec{B})U^{-1}$ by a special unitary matrix $U\in SU(2)$,

$$SU(2)=\{U\in\mathrm{Mat}(2,\mathbb{C})\mid U^\dagger U=I,\ \det U=1\}.$$

In this way, not only vectors but also rotations become represented within the matrix algebra $\mathrm{Mat}(2,\mathbb{C})$. In fact, each rotation $R\in SO(3)$ becomes represented by two matrices $\pm U\in SU(2)$, and we say that $SU(2)$ is a two-fold covering of $SO(3)$:

$$SO(3)\simeq\frac{SU(2)}{\{\pm I\}}.$$

Pauli spinors could also be replaced by square matrices with only the first column being non-zero,

$$\psi=\begin{pmatrix}\psi_1 & 0\\ \psi_2 & 0\end{pmatrix},\quad \psi_1,\psi_2\in\mathbb{C}.$$

Such square matrix spinors form a *left ideal* S of the matrix algebra $\mathrm{Mat}(2,\mathbb{C})$, that is, for $U\in\mathrm{Mat}(2,\mathbb{C})$ and $\psi\in S$ we also have $U\psi\in S$. [5]

The matrix algebra $\mathrm{Mat}(2,\mathbb{C})$ is an isomorphic image of the Clifford algebra $C\ell_3$ of the Euclidean space $\mathbb{R}^3$. Thus, not only vectors in $\mathbb{R}$ and rotations in

5 The left ideal can be written as $S=\mathrm{Mat}(2,\mathbb{C})f$, where $f=\frac{1}{2}(I+\sigma_3)$ is an idempotent satisfying $f^2=f$. The idempotent is primitive and the left ideal is minimal.

$SO(3)$ have representatives in $C\ell_3$, but also spinor spaces or spinor representations of the rotation group $SO(3)$ [6] can be constructed within the Clifford algebra $C\ell_3$. [7]

In the notation of the Clifford algebra $C\ell_3$ we could describe Pauli's achievement by saying that he replaced $\pi^2 = \vec{\pi} \cdot \vec{\pi}$ by $\vec{\pi}^2 = \vec{\pi} \cdot \vec{\pi} + \vec{\pi} \wedge \vec{\pi} = \pi^2 - \hbar e \vec{B}$ and came across the equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}[\pi^2 - \hbar e \vec{B}]\psi - eV\psi$$

where $\vec{B} \in \mathbb{R}^3 \subset C\ell_3$ and $\psi(\vec{r}, t) \in S = C\ell_3 f$, $f = \frac{1}{2}(1+\mathbf{e}_3)$. All the arguments and functions now have values in one algebra, which will facilitate numerical computations.

In this chapter we shall study more closely the Clifford algebra $C\ell_3$ and the spin group $\mathbf{Spin}(3)$, and reformulate once more the Schrödinger-Pauli equation in terms of $C\ell_3$.

4.1 Orthogonal unit vectors, orthonormal basis

The 3-dimensional Euclidean space $\mathbb{R}^3$ has a basis consisting of three orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The *Clifford algebra* $C\ell_3$ of $\mathbb{R}^3$ is the real associative algebra generated by the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfying the relations

$$\begin{aligned} &\mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = 1, \quad \mathbf{e}_3^2 = 1, \\ &\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1, \quad \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_1, \quad \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2. \end{aligned}$$

The Clifford algebra $C\ell_3$ is 8-dimensional with the following basis:

1	the scalar
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	vectors
$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$	bivectors
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	a volume element.

We abbreviate the unit bivectors as $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j$, when $i \neq j$, and the unit oriented volume element as $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. An arbitrary element in $C\ell_3$ is a sum of a scalar, a vector, a bivector and a volume element, and can be written as $\alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$, where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Example. Compute the product $\mathbf{e}_{12}\mathbf{e}_{13}$. By definition $\mathbf{e}_{12}\mathbf{e}_{13} = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3)$

6 Actually, spinor representations are representations of the universal covering group $SU(2) \simeq \mathbf{Spin}(3)$ of $SO(3)$. The spinor representations cannot be reached by tensor methods, as irreducible components of tensor products of antisymmetric powers of $\mathbb{R}^3$.

7 The orthogonal group $O(3)$ also has a non-trivial covering group $\mathbf{Pin}(3)$ residing within $C\ell_3$.

and by associativity $(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3$. Use anticommutativity, $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$, and substitute $\mathbf{e}_1^2 = 1$ to get $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_1^2\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_{23}$. ∎

Imaginary units. The three unit bivectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_2\mathbf{e}_3$ represent unit oriented plane segments as well as generators of rotations in the coordinate planes, and share the basic property of the imaginary unit, $(\mathbf{e}_i\mathbf{e}_j)^2 = -1$ for $i \neq j$. The oriented volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ also shares the basic property of the imaginary unit, $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = -1$, and furthermore it commutes with all the elements in $C\ell_3$. The unit oriented volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ represents the duality operator, which swaps plane segments and line segments orthogonal to the plane segments. ∎

4.2 Matrix representation of $C\ell_3$

The set of 2×2-matrices with complex numbers as entries is denoted by $\mathrm{Mat}(2,\mathbb{C})$. Mostly we shall regard this set as a *real* algebra with scalar multiplication taken over the real numbers in $\mathbb{R}$ although the matrix entries are in the complex field $\mathbb{C}$. The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the multiplication rules

$$\begin{aligned} &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I \quad \text{and} \\ &\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1, \\ &\sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3, \\ &\sigma_2\sigma_3 = i\sigma_1 = -\sigma_3\sigma_2. \end{aligned}$$

They also generate the real algebra $\mathrm{Mat}(2,\mathbb{C})$. The correspondences $\mathbf{e}_1 \simeq \sigma_1$, $\mathbf{e}_2 \simeq \sigma_2$, $\mathbf{e}_3 \simeq \sigma_3$ establish an isomorphism between the real algebras, $C\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$, with the following correspondences of the basis elements:

$\mathrm{Mat}(2,\mathbb{C})$	$C\ell_3$
I	1
$\sigma_1,\ \sigma_2,\ \sigma_3$	$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$
$\sigma_1\sigma_2,\ \sigma_1\sigma_3,\ \sigma_2\sigma_3$	$\mathbf{e}_{12},\ \mathbf{e}_{13},\ \mathbf{e}_{23}$
$\sigma_1\sigma_2\sigma_3$	$\mathbf{e}_{123}$

Note that $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ for $i \neq j$. The essential difference between the Clifford algebra $C\ell_3$ and its matrix image $\mathrm{Mat}(2,\mathbb{C})$ is that in the Clifford algebra $C\ell_3$ we will, in its definition, distinguish a particular subspace, the vector space $\mathbb{R}^3$, in which the square of a vector equals its length squared, that is, $\mathbf{r}^2 = |\mathbf{r}|^2$. No such distinguished subspace has been singled out in the definition of the matrix algebra $\mathrm{Mat}(2, \mathbb{C})$. Instead, we have chosen the traceless Hermitian matrices to represent $\mathbb{R}^3$, and thereby added extra structure to $\mathrm{Mat}(2, \mathbb{C})$. [8]

4.3 The center of $\mathcal{C}\ell_3$

The element $\mathbf{e}_{123}$ commutes with all the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and therefore with every element of $\mathcal{C}\ell_3$. In other words, elements of the form

$$x + y\mathbf{e}_{123} \simeq \begin{pmatrix} x + iy & 0 \\ 0 & x + iy \end{pmatrix}$$

commute with all the elements in $\mathcal{C}\ell_3$. The subalgebra of scalars and 3-vectors

$$\mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3 = \{x + y\mathbf{e}_{123} \mid x, y \in \mathbb{R}\}$$

is the *center* $\mathrm{Cen}(\mathcal{C}\ell_3)$ of $\mathcal{C}\ell_3$, that is, it consists of those elements of $\mathcal{C}\ell_3$ which commute with every element of $\mathcal{C}\ell_3$. Note that $\sigma_1\sigma_2\sigma_3 = iI$. Since $\mathbf{e}_{123}^2 = -1$, the center of $\mathcal{C}\ell_3$ is isomorphic to the complex field $\mathbb{C}$, that is,

$$\mathrm{Cen}(\mathcal{C}\ell_3) = \mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3 \simeq \mathbb{C}.$$

4.4 The even subalgebra $\mathcal{C}\ell_3^+$

The elements 1 and $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_{13} = \mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_{23} = \mathbf{e}_2\mathbf{e}_3$ are called *even*, because they are products of an even number of vectors. The even elements are represented by the following matrices:

$$w + x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12} \simeq \begin{pmatrix} w + iz & ix + y \\ ix - y & w - iz \end{pmatrix}.$$

The even elements form a real subspace

$$\begin{aligned} \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3 &= \{w + x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12} \mid w, x, y, z \in \mathbb{R}\} \\ &\simeq \{wI + xi\sigma_1 + yi\sigma_2 + zi\sigma_3 \mid w, x, y, z \in \mathbb{R}\} \end{aligned}$$

8 We could also have chosen, for the representatives of the anticommuting (and therefore orthogonal) unit vectors in $\mathbb{R}^3$, the following matrices:

$$u_1 = \frac{1}{4}\begin{pmatrix} 3i & 5 \\ 5 & -3i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_3 = \frac{1}{4}\begin{pmatrix} 5 & -3i \\ -3i & -5 \end{pmatrix},$$

that is, $u_1 = \frac{1}{4}(5\sigma_1 + 3\sigma_1\sigma_2)$, $u_2 = \sigma_2$, $u_3 = \frac{1}{4}(5\sigma_3 - 3\sigma_2\sigma_3)$. These matrices are non-Hermitian and satisfy $u_j u_k + u_k u_j = 2\delta_{jk} I$.

which is closed under multiplication. Thus, the subspace $\mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3$ is a subalgebra, called the *even subalgebra* of $\mathcal{C}\ell_3$. We will denote the even subalgebra by even$(\mathcal{C}\ell_3)$ or for short by $\mathcal{C}\ell_3^+$. The even subalgebra is isomorphic to the division ring of quaternions $\mathbb{H}$, as can be seen by the following correspondences:

$\mathbb{H}$	$\mathcal{C}\ell_3^+$
i	$-\mathbf{e}_{23}$
j	$-\mathbf{e}_{31}$
k	$-\mathbf{e}_{12}$

Remark. The Clifford algebra $\mathcal{C}\ell_3$ contains two subalgebras, isomorphic to $\mathbb{C}$ [the center] and $\mathbb{H}$ [the even subalgebra], in such a way that [temporarily we denote these subalgebras by their isomorphic images]

1. $ab = ba$ for $a \in \mathbb{C}$ and $b \in \mathbb{H}$,
2. $\mathcal{C}\ell_3$ is generated as a real algebra by $\mathbb{C}$ and $\mathbb{H}$,
3. $(\dim \mathbb{C})(\dim \mathbb{H}) = \dim \mathcal{C}\ell_3$.

These three observations can be expressed as

$$\mathbb{C} \otimes \mathbb{H} \simeq \mathcal{C}\ell_3.$$

∎

4.5 Involutions of $\mathcal{C}\ell_3$

The Clifford algebra $\mathcal{C}\ell_3$ has three involutions similar to complex conjugation. Take an arbitrary element

$$u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 + \langle u\rangle_3 \quad \text{in} \quad \mathcal{C}\ell_3,$$

written as a sum of a scalar $\langle u\rangle_0$, a vector $\langle u\rangle_1$, a bivector $\langle u\rangle_2$ and a volume element $\langle u\rangle_3$. We introduce the following involutions:

$$\begin{aligned} \hat{u} &= \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2 - \langle u\rangle_3, && \text{grade involution,} \\ \tilde{u} &= \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2 - \langle u\rangle_3, && \text{reversion,} \\ \bar{u} &= \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2 + \langle u\rangle_3, && \text{Clifford-conjugation.} \end{aligned}$$

Clifford-conjugation is a composition of the two other involutions: $\bar{u} = \hat{u}\tilde{\ } = \tilde{u}\hat{\ }$.

The correspondences $\sigma_1 \simeq \mathbf{e}_1$, $\sigma_2 \simeq \mathbf{e}_2$, $\sigma_3 \simeq \mathbf{e}_3$ fix the following representations for the involutions:

$$u \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C},$$

$$\hat{u} \simeq \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}, \qquad \tilde{u} \simeq \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \qquad \bar{u} \simeq \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where the asterisk denotes complex conjugation. We recognize that the reverse $\tilde{u}$ is represented by the Hermitian conjugate $u^\dagger$ and the Clifford-conjugate $\bar{u}$ by the matrix $u^{-1}\det u \in \mathrm{Mat}(2,\mathbb{R})$ [for an invertible u].

The grade involution is an automorphism, that is,

$$\widehat{uv} = \hat{u}\hat{v},$$

while the reversion and the conjugation are anti-automorphisms, that is,

$$\widetilde{uv} = \tilde{v}\tilde{u} \quad \text{and} \quad \overline{uv} = \bar{v}\bar{u}.$$

The grade involution induces the even-odd grading of $\mathcal{C}\ell_3 = \mathcal{C}\ell_3^+ \oplus \mathcal{C}\ell_3^-$.

The reversion can be used to extend the norm from $\mathbb{R}^3$ to all of $\mathcal{C}\ell_3$ by setting

$$|u|^2 = \langle u\tilde{u}\rangle_0.$$

The norm of

$$u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_{12}\mathbf{e}_{12} + u_{13}\mathbf{e}_{13} + u_{23}\mathbf{e}_{23} + u_{123}\mathbf{e}_{123}$$

can be obtained from

$$|u|^2 = |u_0|^2 + |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2 + |u_{123}|^2.$$

The norm satisfies the inequality

$$|uv| \leq \sqrt{2}|u||v| \quad \text{for} \quad u, v \in \mathcal{C}\ell_3.$$

The conjugation can be used to determine the inverse

$$u^{-1} = \frac{\bar{u}}{u\bar{u}}$$

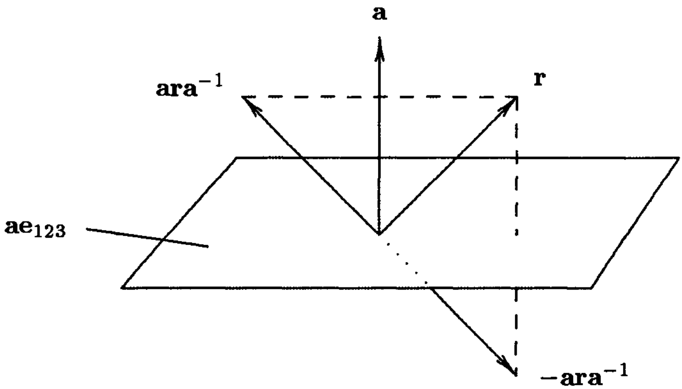
of $u \in \mathcal{C}\ell_3$, $u\bar{u} \neq 0$. The element $u\bar{u} = \bar{u}u$ is in the center $\mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3$ of $\mathcal{C}\ell_3$, so that division by it is unambiguous.

4.6 Reflections and rotations

In the Euclidean space $\mathbb{R}^3$ the vectors $\mathbf{r}$ and $\mathbf{ara}^{-1} = 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a}^{-1} - \mathbf{r}$ are symmetric with respect to the axis $\mathbf{a}$ [use the definition of the Clifford product, $\mathbf{ar} + \mathbf{ra} = 2\mathbf{a}\cdot\mathbf{r}$]. The opposite of $\mathbf{ara}^{-1}$, the vector

$$-\mathbf{ara}^{-1} = \mathbf{r} - 2\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}^2}\mathbf{a},$$

is obtained by reflecting $\mathbf{r}$ across the mirror perpendicular to $\mathbf{a}$ [reflection across the plane $\mathbf{ae}_{123}$].



Two successive reflections in planes perpendicular to $\mathbf{a}$ and $\mathbf{b}$ result in a rotation $\mathbf{r} \to \mathbf{bara}^{-1}\mathbf{b}^{-1}$ around the axis which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Indeed, $\mathbf{r}$ can be decomposed as $\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_{\perp}$ where $\mathbf{r}_{||}$ and $\mathbf{r}_{\perp}$ are parallel and perpendicular, respectively, to the plane of $\mathbf{a}$ and $\mathbf{b}$. The perpendicular component $\mathbf{r}_{\perp}$ remains invariant under both the reflections while the two successive reflections together rotate the parallel component $\mathbf{r}_{||}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ by twice the angle between $\mathbf{a}$ and $\mathbf{b}$.

Consider a vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and the bivector $\mathbf{ae}_{123} = a_1\mathbf{e}_{23} + a_2\mathbf{e}_{31} + a_3\mathbf{e}_{12}$ dual to $\mathbf{a}$. The vector $\mathbf{a}$ has positive square

$$\mathbf{a}^2 = |\mathbf{a}|^2, \quad \text{where} \quad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

but the bivector $\mathbf{ae}_{123}$ has negative square

$$(\mathbf{ae}_{123})^2 = -|\mathbf{a}|^2.$$

It follows that

$$\exp(\mathbf{ae}_{123}) = \cos\alpha + \mathbf{e}_{123}\frac{\mathbf{a}}{\alpha}\sin\alpha$$

where $\alpha = |\mathbf{a}|$. A spatial rotation of the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ around the axis $\mathbf{a}$ by the angle α is given by

$$\mathbf{r} \to a\mathbf{r}a^{-1}, \qquad a = \exp(\tfrac{1}{2}\mathbf{ae}_{123}).$$

The sense of the rotation is clockwise when regarded from the arrow-head of $\mathbf{a}$. The axis of two consecutive rotations around the axes $\mathbf{a}$ and $\mathbf{b}$ is given by the *Rodrigues formula*

$$\mathbf{c}' = \frac{\mathbf{a}' + \mathbf{b}' + \mathbf{a}' \times \mathbf{b}'}{1 - \mathbf{a}' \cdot \mathbf{b}'} \quad \text{where} \quad \mathbf{a}' = \frac{\mathbf{a}}{\alpha}\tan\frac{\alpha}{2}\,.$$

This result is obtained by dividing both sides of the formula

$$\exp(\tfrac{1}{2}\mathbf{c}\mathbf{e}_{123}) = \exp(\tfrac{1}{2}\mathbf{b}\mathbf{e}_{123})\exp(\tfrac{1}{2}\mathbf{a}\mathbf{e}_{123})$$

by their scalar parts and then by inspecting the bivector parts.

4.7 The group Spin(3)

The Clifford algebra $C\ell_3$ of $\mathbb{R}^3$ can be employed to construct the universal covering group for the rotation group $SO(3)$ of $\mathbb{R}^3$. A vector $\mathbf{x} \in \mathbb{R}^3$ can be rotated by the formula

$$\mathbb{R}^3 \to \mathbb{R}^3, \ \mathbf{x} \to \rho(s)\mathbf{x} = s\mathbf{x}s^{-1}$$

where s is an element of the group

$$\mathbf{Spin}(3) = \{s \in C\ell_3 \mid \tilde{s}s = 1, \ \bar{s}s = 1\}.$$

The group **Spin**(3), called the *spin group*, is a two-fold covering group of the rotation group $SO(3)$.

In the matrix formulation provided by the Pauli spin matrices, the spin group **Spin**(3) has an isomorphic image, the special unitary group

$$SU(2) = \{s \in \mathrm{Mat}(2,\mathbb{C}) \mid s^\dagger s = I, \ \det s = 1\}.$$

For an element $s \in SU(2)$ the function $\mathbf{x} \to \rho(s)\mathbf{x} = s\mathbf{x}s^\dagger$ is a rotation of the Euclidean space of traceless Hermitian matrices,

$$\{\mathbf{x} \in \mathrm{Mat}(2,\mathbb{C}) \mid \mathrm{trace}(\mathbf{x}) = 0, \ \mathbf{x}^\dagger = \mathbf{x}\} \simeq \mathbb{R}^3.$$

Every element in $SO(3)$ can be represented by a matrix in $SU(2)$. There are two matrices s and $-s$ in $SU(2)$ representing the same rotation $R = \rho(\pm s) \in SO(3)$. In other words, the group homomorphism $\rho : \mathbf{Spin}(3) \to SO(3)$ is surjective with kernel $\{\pm 1\}$. This can be depicted by a sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{Spin}(3) \xrightarrow{\rho} SO(3) \longrightarrow 1$$

which is exact, that is, the image of a homomorphism coincides with the kernel of the successive homomorphism.

The spin group **Spin**(3) is a universal cover of the rotation group $SO(3)$, that is, the Lie group **Spin**(3) is simply connected. [9] The group $SO(3)$ is doubly connected. [10]

9 A Lie group is simply connected if it is connected and every loop in the group can be shrunk to a point.

10 Rotations in $SO(3)$ can be represented by vectors $\mathbf{a} \in \mathbb{R}^3$, $|\mathbf{a}| \leq \pi$. Each rotation, $|\mathbf{a}| < \pi$, has a unique representative, and each half-turn, $|\mathbf{a}| = \pi$, is represented twice, $\pm\mathbf{a}$. A loop connecting the identity and a half-turn does not shrink to a point.

4.8 Pauli spinors

In the non-relativistic theory of the spinning electron one considers column matrices, the *Pauli spinors*

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \quad \text{where} \quad \psi_1, \psi_2 \in \mathbb{C}.$$

An isomorphic complex linear space is obtained if one replaces Pauli spinors by the *square matrix spinors*

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}$$

where only the first column is non-zero. The fact that only the first column is non-zero can be expressed as

$$\psi \in \mathrm{Mat}(2,\mathbb{C})f \quad \text{where} \quad f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall regard the correspondences $\mathbf{e}_1 \simeq \sigma_1$, $\mathbf{e}_2 \simeq \sigma_2$, $\mathbf{e}_3 \simeq \sigma_3$ as an identification between $C\ell_3$ and $\mathrm{Mat}(2,\mathbb{C})$. If we multiply $\psi \in \mathrm{Mat}(2,\mathbb{C})f$ on the left by an arbitrary element $u \in C\ell_3 = \mathrm{Mat}(2,\mathbb{C})$, then the result is also of the same type:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}.$$

Such matrices, with only the first column being non-zero, form a *left ideal* S of $C\ell_3$, that is,

$$u\psi \in S \quad \text{for all} \quad u \in C\ell_3 \quad \text{and} \quad \psi \in S \subset C\ell_3.$$

This left ideal S of $C\ell_3$ contains no left ideal other than S itself and the zero ideal $\{0\}$. Such a left ideal is called *minimal* in $C\ell_3$.

As a real linear space, S has a basis $\{f_0, f_1, f_2, f_3\}$ where

$$\begin{aligned}
f_0 &= \tfrac{1}{2}(1 + \mathbf{e}_3) &&\simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
f_1 &= \tfrac{1}{2}(\mathbf{e}_{23} + \mathbf{e}_2) &&\simeq \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \\
f_2 &= \tfrac{1}{2}(\mathbf{e}_{31} - \mathbf{e}_1) &&\simeq \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\
f_3 &= \tfrac{1}{2}(\mathbf{e}_{12} + \mathbf{e}_{123}) &&\simeq \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

The element $f = f_0$ is an *idempotent*, that is, $f^2 = f$.

The subset

$$\mathbb{F} = fC\ell_3 f \simeq \left\{ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \middle| \, c \in \mathbb{C} \right\}$$

of $C\ell_3$ is a subring with unity f, that is, $af = fa$ for $a \in \mathbb{F}$. None of the elements of $\mathbb{F}$ is invertible as an element of $C\ell_3$, but for each non-zero $a \in \mathbb{F}$ there is a unique $b \in \mathbb{F}$ such that $ab = f$. Thus, $\mathbb{F}$ is a *division ring* with unity f [this follows from the idempotent f being *primitive* in $C\ell_3$]. As a 2-dimensional real division algebra $\mathbb{F}$ must be isomorphic to $\mathbb{C}$. The isomorphism $\mathbb{F} \simeq \mathbb{C}$ is seen by the equation $f_3^2 = -f_0$ relating the basis elements $\{f_0, f_3\}$ of the real algebra $\mathbb{F}$.

Comment. The multiplication of an element ψ of the real linear space S on the left by an arbitrary even element $u \in C\ell_3^+$, expressed in coordinate form in the basis $\{f_0, f_1, f_2, f_3\}$,

$$u\psi = (u_0 + u_1 \mathbf{e}_{23} + u_2 \mathbf{e}_{31} + u_3 \mathbf{e}_{23})(\psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_3 f_3),$$

corresponds to the matrix multiplication

$$u\psi \simeq \begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

The square matrices corresponding to the left multiplication by even elements constitute a subring of $\mathrm{Mat}(4, \mathbb{R})$; this subring is an isomorphic image of the quaternion ring $\mathbb{H}$. ∎

The minimal left ideal

$$S = C\ell_3 f \simeq \left\{ \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \middle| \, \psi_1, \psi_2 \in \mathbb{C} \right\}$$

has a natural right $\mathbb{F}$-linear structure defined by

$$S \times \mathbb{F} \to S, \ (\psi, \lambda) \to \psi\lambda.$$

We shall provide the minimal left ideal S with this right $\mathbb{F}$-linear structure, and call it a *spinor space*. [11]

The map $C\ell_3 \to \mathrm{End}_{\mathbb{F}} S$, $u \to \tau(u)$, where $\tau(u)$ is defined by the relation $\tau(u)\psi = u\psi$, is a real algebra isomorphism. Employing the basis $\{f_0, -f_2\}$ for the $\mathbb{F}$-linear space S, the elements $\tau(\mathbf{e}_1), \tau(\mathbf{e}_2), \tau(\mathbf{e}_3)$ will be represented by the matrices $\sigma_1, \sigma_2, \sigma_3$. In this way the Pauli matrices are reproduced.

11 Note that multiplying a matrix ψ in S, a left ideal, on the left by $\lambda \in \mathbb{F}$ does not result in a left $\mathbb{F}$-linear structure.

There is a natural way to introduce scalar products on the spinor space $S \subset \mathcal{C}\ell_3$. First, note that for all $\psi, \varphi \in S$ the product

$$\tilde{\psi}\varphi \simeq \begin{pmatrix} \psi_1^* & \psi_2^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1^*\varphi_1 + \psi_2^*\varphi_2 & 0 \\ 0 & 0 \end{pmatrix}$$

falls in the division ring $\mathbb{F}$ ($z \to z^*$ means complex conjugation). To show that the map

$$S \times S \to \mathbb{F}, \ (\psi, \varphi) \to \tilde{\psi}\varphi$$

defines a scalar product we only have to verify that the reversion $\psi \to \tilde{\psi}$ is a right-to-left $\mathbb{F}$-semilinear map. For all $\psi \in S$, $\lambda \in \mathbb{F}$ we have $(\psi\lambda)\tilde{} = \tilde{\lambda}\tilde{\psi}$ where the map $\lambda \to \tilde{\lambda}$ is an anti-involution of the division algebra $\mathbb{F}$ (actually complex conjugation).

Multiplying a spinor $\psi \in S \subset \mathcal{C}\ell_3$ by an element $s \in \mathcal{C}\ell_3$ is a right $\mathbb{F}$-linear transformation $S \to S$, $\psi \to s\psi$. The automorphism group of the scalar product is formed by those right $\mathbb{F}$-linear transformations which preserve the scalar product, that is,

$$(s\psi)\tilde{}(s\varphi) = \tilde{\psi}\varphi \quad \text{for all} \quad \psi, \varphi \in S.$$

The automorphism group of the scalar product $\tilde{\psi}\varphi$ is seen to be the group $\{s \in \mathcal{C}\ell_3 \mid \tilde{s}s = 1\}$ which is isomorphic to the group of unitary 2×2-matrices,

$$U(2) = \{s \in \mathrm{Mat}(2, \mathbb{C}) \mid s^\dagger s = I\}.$$

We can also use the Clifford conjugate $u \to \bar{u}$ of $\mathcal{C}\ell_3$ to introduce a scalar product for spinors. In this case, the element

$$\bar{\psi}\varphi \simeq \begin{pmatrix} 0 & 0 \\ -\psi_2 & \psi_1 \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \psi_1\varphi_2 - \psi_2\varphi_1 & 0 \end{pmatrix}$$

does not appear in the division ring $\mathbb{F} = f\mathcal{C}\ell_3 f$. However, we can find an invertible element $a \in \mathcal{C}\ell_3$ so that $a\bar{\psi}\varphi \in \mathbb{F}$, e.g. $a = \mathbf{e}_1$ or $a = \mathbf{e}_{31}$. The map

$$S \times S \to \mathbb{F}, \ (\psi, \varphi) \to a\bar{\psi}\varphi$$

defines a scalar product. Writing

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we find that $a\bar{\psi}\varphi \simeq \tau(\psi)^{\mathsf{T}} J \tau(\varphi)$. Hence, the automorphism group $\{s \in \mathcal{C}\ell_3 \mid \bar{s}s = 1\}$ of the scalar product $a\bar{\psi}\varphi$ is the group of symplectic 2×2-matrices,

$$Sp(2, \mathbb{C}) = \{s \in \mathrm{Mat}(2, \mathbb{C}) \mid s^{\mathsf{T}} J s = J\}.$$

4.9 Spinor operators

Up till now spinors have been objects which have been operated upon. Next we will replace such passive spinors by active spinor operators. Instead of spinors

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \in C\ell_3 f$$

in minimal left ideals we will consider the following even elements:

$$\Psi = 2\,\mathrm{even}(\psi) = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \in C\ell_3^+,$$

also computed as $\Psi = \psi + \hat{\psi}$ for $\psi \in C\ell_3 f$. Classically, the expectation values of the components of the spin have been determined in terms of the column spinor $\psi \in \mathbb{C}^2$ by computing the following three real numbers:

$$s_1 = \psi^\dagger \sigma_1 \psi, \quad s_2 = \psi^\dagger \sigma_2 \psi, \quad s_3 = \psi^\dagger \sigma_3 \psi.$$

In terms of $\psi \in C\ell_3 f$ this computation could be repeated as

$$s_1 = 2\langle \psi \mathbf{e}_1 \tilde{\psi} \rangle_0, \quad s_2 = 2\langle \psi \mathbf{e}_2 \tilde{\psi} \rangle_0, \quad s_3 = 2\langle \psi \mathbf{e}_3 \tilde{\psi} \rangle_0.$$

However, in terms of $\Psi \in C\ell_3^+$ we may compute $\mathbf{s} = s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3$ directly as

$$\mathbf{s} = \Psi \mathbf{e}_3 \tilde{\Psi}.$$

Since Ψ acts here like an operator, we call it a *spinor operator*. It should be emphasized that not only did we get all the components of the spin vector $\mathbf{s}$ at one stroke, but we also got the entity $\mathbf{s}$ as a whole.

Remark. The mapping $C\ell_3^+ \to \mathbb{R}^3$, $\Psi \to \Psi\sigma_3\Psi^\dagger = \Psi\mathbf{e}_3\tilde{\Psi}$ is the KS-transformation (introduced by Kustaanheimo & Stiefel 1965) for spinor regularization of Kepler motion, and its restriction to norm-one spinor operators Ψ satisfying $\Psi\tilde{\Psi} = 1$ (or equivalently $\Psi\Psi^\dagger = I$) results in a Hopf fibration $S^3 \to S^2$ (the matrix $\Psi\sigma_3\Psi^\dagger$ is both unitary and involutory and represents a reflection of the spinor space with axis ψ).

The above mapping should not be confused with the 'Cartan map', see Cartan 1966 p. 41 and Keller & Rodríguez-Romo 1991 p. 1591. A 'Cartan map' $\mathbb{C}^2 \times \mathbb{C}^2 \to C\ell_3$, $(\psi, \varphi) \to 2\psi\mathbf{e}_1\bar{\varphi}$, where $\mathbb{C}^2 = C\ell_3 f$, sends a pair of square matrix spinors to a complex 4-vector $x_0 + \mathbf{x}$,

$$x_0 = -(\psi_1\varphi_2 - \psi_2\varphi_1), \quad \mathbf{x} = \begin{pmatrix} \psi_1\varphi_1 - \psi_2\varphi_2 \\ i(\psi_1\varphi_1 + \psi_2\varphi_2) \\ -(\psi_1\varphi_2 + \psi_2\varphi_1) \end{pmatrix}.$$

When $\psi = \varphi$, $\mathbf{x}^2 = 0$. ∎

Note also that $\mathrm{trace}(\psi\psi^\dagger) = 2\langle \psi\tilde{\psi} \rangle_0 = \Psi\tilde{\Psi}$ which equals $\Psi\bar{\Psi} = \det(\Psi)$.

In operator form the Schrödinger-Pauli equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\pi^2\Psi - \frac{\hbar e}{2m}\vec{B}\Psi\mathbf{e}_3 - eV\Psi$$

shows explicitly the quantization direction $\mathbf{e}_3$ of the spin. The explicit occurrence of $\mathbf{e}_3$ is due to the injection $\mathbb{C}^2 \to \mathcal{C}\ell_3 f$, $\psi \to \Psi$; technically $2\,\text{even}(\vec{B}\psi) = \vec{B}\Psi\mathbf{e}_3$. If we rotate the system $90°$ around the y-axis, counter-clockwise as seen from the positive y-axis, then vectors and spinors transform to

$$\vec{B}' = u\vec{B}u^{-1} \quad \text{and} \quad \Psi' = u\Psi \quad \text{where} \quad u = \exp(\frac{\pi}{4}\mathbf{e}_{13}),$$

and the Pauli equation transforms to

$$i\hbar\frac{\partial\Psi'}{\partial t} = \frac{1}{2m}\pi'^2\Psi' - \frac{\hbar e}{2m}\vec{B}'\Psi'\mathbf{e}_3 - eV\Psi'.$$

If this equation is multiplied on the right by u^{-1}, then $\mathbf{e}_3$ goes to $\mathbf{e}_1 = u\mathbf{e}_3u^{-1}$, and the equation looks like

$$i\hbar\frac{\partial\Psi''}{\partial t} = \frac{1}{2m}\pi'^2\Psi'' - \frac{\hbar e}{2m}\vec{B}'\Psi''\mathbf{e}_1 - eV\Psi'',$$

where $\Psi'' = u\Psi u^{-1}$. Both the transformation laws give the same values for observables, that is, $\Psi'\mathbf{e}_3\tilde{\Psi}' = \Psi''\mathbf{e}_1\tilde{\Psi}''$.

Exercises

1. Compute the square of $\mathbf{a} + \mathbf{b}\mathbf{e}_{123}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
2. Compute p^2, q^2 and pq for $p = \frac{1}{2}(1 + \mathbf{e}_3)$ and $q = \frac{1}{2}(1 - \mathbf{e}_3)$.
3. Compute the squares of $\frac{1}{2}(1 + \mathbf{e}_3) \pm \frac{1}{2}(1 - \mathbf{e}_3)\mathbf{e}_{12}$.
4. Find all the four square roots of $\cos\varphi + \mathbf{e}_{12}\sin\varphi$. Hint: $\mathbf{e}_{12}\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_{12}$.
5. Find the exponentials of $\pm\frac{\pi}{2}(1 - \mathbf{e}_3)\mathbf{e}_{12}$. Hint: $\mathbf{e}_{12}$ and $\mathbf{e}_{123}$ commute [or $q = \frac{1}{2}(1 - \mathbf{e}_3)$ is an idempotent satisfying $q^2 = q$].
6. Let $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$ [$\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$]. Compute $u\bar{u}$.
7. Find the inverse of $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$. Hint: $u\bar{u}$ is of the form $x + y\mathbf{e}_{123}$, $x, y \in \mathbb{R}$.
8. Find the exponential of $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$. Hint: compute $(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2$.
9. Show that each non-zero even element in $\mathcal{C}\ell_3^+$ is invertible.
10. Show that $u\tilde{u} \in \mathbb{R} \oplus \mathbb{R}^3$ for all $u \in \mathcal{C}\ell_3$.
11. Show that $|u\mathbf{a}\tilde{u}| = |u|^2|\mathbf{a}|$ for $\mathbf{a} \in \mathbb{R}^3$, $u \in \mathbb{R} \oplus \bigwedge^2\mathbb{R}^3$.
12. Show that the norm on $\mathcal{C}\ell_3$, defined by $|u|^2 = \langle u\tilde{u}\rangle_0$, agrees with the

norm given by $|u|^2 = <u, u>$ where the symmetric bilinear product is determined by

$$<\alpha, \beta> = \alpha\beta \quad \text{for} \quad \alpha, \beta \in \mathbb{R},$$
$$<\mathbf{a}, \mathbf{b}> = \mathbf{a} \cdot \mathbf{b} \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$$

and by

$$<\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_k> = \begin{vmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \ldots & \mathbf{x}_1 \cdot \mathbf{y}_k \\ \vdots & \ddots & \vdots \\ \mathbf{x}_k \cdot \mathbf{y}_1 & \ldots & \mathbf{x}_k \cdot \mathbf{y}_k \end{vmatrix}$$

in $\bigwedge^k \mathbb{R}^3$, $k \geq 2$. [One also needs to assume orthogonality of the components in $C\ell_3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3$.]

13. Show that the reflection across the plane of the bivector $\mathbf{A}$ is obtained by $\mathbf{r} \to \mathbf{r}' = -\mathbf{A}\mathbf{r}\mathbf{A}^{-1}$.
14. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. Compute $\langle \mathbf{xyz} \rangle_1$ and $\langle \mathbf{xyz} \rangle_3$. Hint: use reversion.

Solutions

1. $(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2 = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} + 2(\mathbf{a} \cdot \mathbf{b})\mathbf{e}_{123}$.
2. $p^2 = p$ and $q^2 = q$, that is, p and q are idempotents; and $pq = 0$ [and so there are zero-divisors in the Clifford algebra $C\ell_3$].
3. $\mathbf{e}_3$ [this shows that vectors can have square roots].
4. $\pm(\cos \frac{\varphi}{2} + \mathbf{e}_{12} \sin \frac{\varphi}{2})$, $\pm\mathbf{e}_3(\cos \frac{\varphi}{2} + \mathbf{e}_{12} \sin \frac{\varphi}{2})$.
5. $\mathbf{e}_3$ [this shows that vectors also have logarithms].
6. $\alpha^2 - \beta^2 - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2(\alpha\beta - \mathbf{a} \cdot \mathbf{b})\mathbf{e}_{123}$.
8. Denote $r = \sqrt{(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2} \in \mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3$, $v = (\mathbf{a} + \mathbf{b}\mathbf{e}_{123})/r$, $v^2 = 1$. Then $\exp(u) = \exp(\alpha + \beta\mathbf{e}_{123})[\frac{1}{2}(1 + v)\exp(r) + \frac{1}{2}(1 - v)\exp(-r)]$ when $r \neq 0$. When $r = 0$: $\exp(u) = \exp(\alpha + \beta\mathbf{e}_{123})(1 + \mathbf{a} + \mathbf{b}\mathbf{e}_{123})$.
10. $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$, $u\tilde{u} = \alpha^2 + \beta^2 + \mathbf{a}^2 + \mathbf{b}^2 + 2(\alpha\mathbf{a} + \beta\mathbf{b} + \mathbf{a} \times \mathbf{b})$ which is in $\mathbb{R} \oplus \mathbb{R}^3$. Direct proof:

$$(u\tilde{u})\tilde{} = \tilde{\tilde{u}}\tilde{u} = u\tilde{u}$$

which implies $u\tilde{u} \in \mathbb{R} \oplus \mathbb{R}^3$, since the reversion sends bivectors and 3-vectors to their opposites.

13. Decompose $\mathbf{r}$ into components parallel, $\mathbf{r}_{\|}$, and perpendicular, $\mathbf{r}_{\perp}$, to $\mathbf{A}$, and note that $\mathbf{A}$ anticommutes with vectors in its plane,
$\mathbf{A}(\mathbf{r}_{\|} + \mathbf{r}_{\perp}) = (-\mathbf{r}_{\|} + \mathbf{r}_{\perp})\mathbf{A}$. Then
$\mathbf{A}(\mathbf{r}_{\|} + \mathbf{r}_{\perp})\mathbf{A}^{-1} = (-\mathbf{r}_{\|} + \mathbf{r}_{\perp})\mathbf{A}\mathbf{A}^{-1} = -\mathbf{r}'$.
14. First, $(\mathbf{xyz})\tilde{} = \mathbf{zyx}$ and $(\mathbf{xyz})\tilde{} = \langle \mathbf{xyz} \rangle_1 - \langle \mathbf{xyz} \rangle_3$. Therefore,

$\langle \mathbf{xyz} \rangle_1 = \frac{1}{2}(\mathbf{xyz} + \mathbf{zyx})$ and $\langle \mathbf{xyz} \rangle_3 = \frac{1}{2}(\mathbf{xyz} - \mathbf{zyx})$, and also
$\langle \mathbf{xyz} \rangle_1 = (\mathbf{y} \cdot \mathbf{z})\mathbf{x} - (\mathbf{z} \cdot \mathbf{x})\mathbf{y} + (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ and $\langle \mathbf{xyz} \rangle_3 = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$.

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5

Quaternions

We saw in the chapter on *Complex Numbers* that it is convenient to use the real algebra of complex numbers $\mathbb{C}$ to represent the rotation group $SO(2)$ of the plane $\mathbb{R}^2$. In this chapter we shall study rotations of the 3-dimensional space $\mathbb{R}^3$. The composition of spatial rotations is no longer commutative, and we need a non-commutative multiplication to represent the rotation group $SO(3)$. This can be done within the real algebra of 3×3-matrices $\mathrm{Mat}(3,\mathbb{R})$, or by the real algebra of **quaternions**, $\mathbb{H}$, invented by Hamilton.

The complex plane $\mathbb{C}$ is a real linear space $\mathbb{R}^2$, and multiplication by a complex number $c = a + ib$, that is, the map $\mathbb{C} \to \mathbb{C}$, $z \to cz$, may be regarded as a real linear map with matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ operating on $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$. The complex plane is also a real quadratic space $\mathbb{R}^{2,0}$, in short $\mathbb{R}^2$, with a quadratic form

$$\mathbb{C} \to \mathbb{R},\ z = x + iy \to z\bar{z} = x^2 + y^2,$$

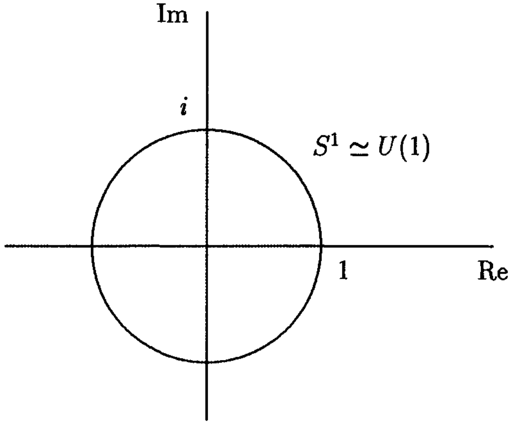
and norm $|z| = \sqrt{z\bar{z}}$. Multiplication of complex numbers preserves the norm, that is, $|cz| = |c||z|$ for all $c, z \in \mathbb{C}$, and so multiplication by c is a rotation of $\mathbb{R}^2$ if, and only if, $|c| = 1$. Conversely, any rotation of $\mathbb{R}^2$ can be represented by a unit complex number c, $|c| = 1$, in $\mathbb{C}$. The unit complex numbers form a group

$$U(1) = \{c \in \mathbb{C} \mid c\bar{c} = 1\},$$

called the *unitary group*, which is isomorphic to the rotation group $SO(2) = \{U \in \mathrm{Mat}(2,\mathbb{R}) \mid U^{\top}U = I,\ \det U = 1\}$, that is, $U(1) \simeq SO(2)$. The unitary group $U(1)$ can be visualized as the *unit circle*

$$S^1 = \{x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$$

of the complex plane $\mathbb{C}$.



Similarly, the algebra of quaternions $\mathbb{H}$ may be used to represent rotations of the 3-dimensional space $\mathbb{R}^3$. It will turn out that quaternions are also convenient to represent the rotations of the 4-dimensional space $\mathbb{R}^4$.

Quaternions as hypercomplex numbers

Quaternions are generalized complex numbers of the form $q = w + ix + jy + kz$ where w, x, y, z are real numbers and the generalized imaginary units i, j, k satisfy the following multiplication rules:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

Note that the multiplication is by definition **non-commutative**. One can show that quaternion multiplication is *associative*. The above multiplication rules can be condensed into the following form:

$$i^2 = j^2 = k^2 = ijk = -1$$

where in the last identity we have omitted parentheses and thereby tacitly assumed associativity.

The generalized imaginary units will be denoted either by i, j, k or by $\mathbf{i}, \mathbf{j}, \mathbf{k}$. They have two different roles: they act as generators of

rotations, that is, they are bivectors, or

translations, that is, they are vectors.

This distinction is not clear-cut since bivectors are dual to vectors in $\mathbb{R}^3$.

5.1 Pure part and cross product

A quaternion $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is a sum of a scalar and a vector, called the *real part*, $\mathrm{Re}(q) = w \in \mathbb{R}$, and the *pure part*, $\mathrm{Pu}(q) = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \in \mathbb{R}^3$. The quaternions form a 4-dimensional real linear space $\mathbb{H}$ which contains the real axis $\mathbb{R}$ and a 3-dimensional real linear space $\mathbb{R}^3$ so that $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$. We denote the pure part also by a boldface letter so that $q = q_0 + \mathbf{q}$ where $q_0 \in \mathbb{R}$ and $\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{R}^3$. The real linear space $\mathbb{R} \oplus \mathbb{R}^3$ with the quaternion product is an associative algebra over $\mathbb{R}$ called the *quaternion algebra* $\mathbb{H}$. The product of two quaternions $a = a_0 + \mathbf{a}$ and $b = b_0 + \mathbf{b}$ can be written as

$$ab = a_0b_0 - \mathbf{a} \cdot \mathbf{b} + a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b}.$$

A quaternion $q = q_0 + \mathbf{q}$ is *pure* if its real part vanishes, $q_0 = 0$, so that $q = \mathbf{q} \in \mathbb{R}^3$. A product of two pure quaternions $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ and $\mathbf{b} = \mathbf{i}b_1 + \mathbf{j}b_2 + \mathbf{k}b_3$ is a sum of a real number and a pure quaternion:

$$\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

where we recognize the scalar product $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and the cross product $\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2b_3 - a_3b_2) + \mathbf{j}(a_3b_1 - a_1b_3) + \mathbf{k}(a_1b_2 - a_2b_1)$.

The vector space $\mathbb{R}^3$ with the cross product $\mathbf{a} \times \mathbf{b}$ is a real algebra, that is, it is a real linear space with a bilinear map

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (\mathbf{a}, \mathbf{b}) \to \mathbf{a} \times \mathbf{b}.$$

The cross product satisfies two rules

$$\begin{aligned}
&\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \\
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0,
\end{aligned}$$

the latter being called the Jacobi identity; this makes $\mathbb{R}^3$ with the cross product a *Lie algebra*. In particular, the cross product is not associative, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

We can reobtain the cross product of two pure quaternions $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ as the pure part of their quaternion product: $\mathbf{a} \times \mathbf{b} = \mathrm{Pu}(\mathbf{ab})$.

5.2 Quaternion conjugate, norm and inverse

The conjugate $\bar{q}$ of a quaternion $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is obtained by changing the sign of the pure part:

$$\bar{q} = w - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z.$$

We shall also refer to $\bar{q}$ as the *quaternion conjugate* of q. The conjugation is an anti-automorphism of $\mathbb{H}$; $\overline{ab} = \bar{b}\bar{a}$ for $a, b \in \mathbb{H}$.

A quaternion q multiplied by its conjugate $\bar{q}$ results in a real number $q\bar{q} = w^2 + x^2 + y^2 + z^2$ called the square norm of $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. The *norm* $|q|$ of q is given by $|q| = \sqrt{q\bar{q}}$ so that

$$|w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

The norm of a product of two quaternions a and b is the product of their norms – as an equation, $|ab| = |a||b|$ for $a, b \in \mathbb{H}$ – which turns $\mathbb{H}$ into a normed algebra.

The *inverse* q^{-1} of a non-zero quaternion q is obtained by $q^{-1} = \bar{q}/|q|^2$ or more explicitly by

$$\frac{1}{w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z} = \frac{w - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z}{w^2 + x^2 + y^2 + z^2}.$$

In particular, $ab = 0$ implies $a = 0$ or $b = 0$, which means that the quaternion algebra is a division algebra (or that the ring of quaternions is a division ring).

5.3 The center of $\mathbb{H}$

The set of those elements in $\mathbb{H}$ which commute with every element of $\mathbb{H}$ forms the *center* of $\mathbb{H}$,

$$\mathrm{Cen}(\mathbb{H}) = \{w \in \mathbb{H} \mid wq = qw \text{ for all } q \in \mathbb{H}\}.$$

The center is of course closed under multiplication. The center of the division ring $\mathbb{H}$ is isomorphic to the field of real numbers $\mathbb{R}$. In contrast to the case of the complex field $\mathbb{C}$, the real axis in $\mathbb{H}$ is the unique subfield which is the center of $\mathbb{H}$.

5.4 Rotations in three dimensions

Take a pure quaternion or a vector

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \in \mathbb{R}^3, \quad \text{where} \quad \mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3,$$

of length $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. For a non-zero quaternion $a \in \mathbb{H}$, the expression $a\mathbf{r}a^{-1}$ is again a pure quaternion with the same length, that is,

$$a\mathbf{r}a^{-1} \in \mathbb{R}^3 \quad \text{and} \quad |a\mathbf{r}a^{-1}| = |\mathbf{r}|.$$

In other words, the mapping

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{r} \to a\mathbf{r}a^{-1}$$

is a rotation of the quadratic space of pure quaternions $\mathbb{R}^3$. Each rotation in $SO(3) = \{U \in \mathrm{Mat}(3,\mathbb{R}) \mid U^{\mathsf{T}}U = I,\ \det U = 1\}$ can be so represented, and there are two unit quaternions a and $-a$ representing the same rotation, $a\mathbf{r}a^{-1} = (-a)\mathbf{r}(-a)^{-1}$. In other words, the sphere of unit quaternions,

$$S^3 = \{q \in \mathbb{H} \mid |q| = 1\},$$

is a two-fold covering group of $SO(3)$, that is, $SO(3) \simeq S^3/\{\pm 1\}$.

A rotation has three parameters in dimension 3. In other words, $SO(3)$ and S^3 are 3-dimensional manifolds. The three parameters are the angle of rotation and the two direction cosines of the axis of rotation.

To find the axis of this rotation we take a unit quaternion a, $|a| = 1$, and write it in the form $a = e^{\mathbf{a}/2}$ where $\mathbf{a} \in \mathbb{R}^3$. Note that

$$e^{\mathbf{a}/2} = \cos\frac{\alpha}{2} + \frac{\mathbf{a}}{\alpha}\sin\frac{\alpha}{2}$$

where $\alpha = |\mathbf{a}|$. The rotation $\mathbf{r} \to a\mathbf{r}a^{-1}$ turns $\mathbf{r}$ about the axis $\mathbf{a}$ by the angle α. The sense of the rotation is counter-clockwise when regarded from the arrow-head of $\mathbf{a}$.

The composite of two consecutive rotations, first around $\mathbf{a}$ by the angle $\alpha = |\mathbf{a}|$ and then around $\mathbf{b}$ by the angle $\beta = |\mathbf{b}|$, is again a rotation around some axis, say $\mathbf{c}$. The axis of the composite rotation can be found by inspection of the real and pure parts of the formula $e^{\mathbf{c}/2} = e^{\mathbf{b}/2}e^{\mathbf{a}/2}$. Divide both sides by their real parts and substitute

$$\mathbf{c}' = \frac{\mathbf{c}}{\gamma}\tan\frac{\gamma}{2}, \quad \text{where} \quad \gamma = |\mathbf{c}|,$$

to obtain the *Rodrigues formula*

$$\mathbf{c}' = \frac{\mathbf{a}' + \mathbf{b}' - \mathbf{a}' \times \mathbf{b}'}{1 - \mathbf{a}' \cdot \mathbf{b}'}.$$

5.5 Rotations in four dimensions

The mapping $\mathbb{H} \to \mathbb{H}$, $q \to aqb^{-1}$, where $a, b \in \mathbb{H}$ are unit quaternions $|a| = |b| = 1$, is a rotation of the 4-dimensional space $\mathbb{R}^4 = \mathbb{H}$. In other words, the real linear mapping

$$\mathbb{H} \to \mathbb{H}, \quad q \to aqb^{-1}, \quad \text{where} \quad a, b \in \mathbb{H} \quad \text{and} \quad |a| = |b| = 1,$$

is a rotation of $\mathbb{R}^4$. Each rotation in $SO(4)$ can be so represented, and there are two elements (a, b) and $(-a, -b)$ in $S^3 \times S^3$ representing the same rotation, that is, $aqb^{-1} = (-a)q(-b)^{-1}$. In other words, the group $S^3 \times S^3$ is a two-fold covering group of $SO(4)$, that is,

$$SO(4) \simeq \frac{S^3 \times S^3}{\{(1,1), (-1,-1)\}}.$$

A rotation in dimension 4 can be represented by a pair of unit quaternions, and so it has six parameters, in other words, $\dim SO(4) = \dim(S^3 \times S^3) = 6$. A rotation has two completely orthogonal invariant planes; both the invariant planes can turn arbitrarily; this takes two parameters. Fixing a plane in $\mathbb{R}^4$ takes the remaining four parameters: three parameters for a unit vector in S^3, plus two parameters for another orthogonal unit vector in S^2, minus one parameter for rotating the pairs of such vectors in the plane.

5.6 Matrix representation of quaternion multiplication

The product of two quaternions $q = w+\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$ and $u = u_0+\mathbf{i}u_1+\mathbf{j}u_2+\mathbf{k}u_3$ can be represented by matrix multiplication:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where $qu = v$. Swapping the multiplication to the right, that is, $uq = v'$, gives a partially transformed matrix:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_0' \\ v_1' \\ v_2' \\ v_3' \end{pmatrix}.$$

Let us denote the above matrices respectively by L_q and R_q, that is,

$$L_q(u) = qu \ (= v) \quad \text{and} \quad R_q(u) = uq \ (= v').$$

We find that [1]

$$L_\mathbf{i} L_\mathbf{j} L_\mathbf{k} = -I \quad \text{and} \quad R_\mathbf{i} R_\mathbf{j} R_\mathbf{k} = I.$$

The sets $\{L_q \in \mathrm{Mat}(4,\mathbb{R}) \mid q \in \mathbb{H}\}$ and $\{R_q \in \mathrm{Mat}(4,\mathbb{R}) \mid q \in \mathbb{H}\}$ form two subalgebras of $\mathrm{Mat}(4,\mathbb{R})$, both isomorphic to $\mathbb{H}$. For two arbitrary quaternions $a, b \in \mathbb{H}$ these two matrix representatives commute, that is, $L_a R_b = R_b L_a$. Any real 4×4-matrix is a linear combination of matrices of the form $L_a R_b$. The above observations together with $(\dim \mathbb{H})^2 = \dim \mathrm{Mat}(4,\mathbb{R})$ imply that

$$\mathrm{Mat}(4,\mathbb{R}) \simeq \mathbb{H} \otimes \mathbb{H},$$

or more informatively $\mathrm{Mat}(4,\mathbb{R}) = \mathbb{H} \otimes \mathbb{H}^*$. [2]

1 Note that $R_\mathbf{i}^\top R_\mathbf{j}^\top R_\mathbf{k}^\top = -I$.

2 For unit quaternions $a, b \in \mathbb{H}$ such that $|a| = |b| = 1$ we may choose $L_a \in Q$ and $R_b \in Q^*$ or alternatively $L_a \in Q^*$ and $R_b \in Q$. For a discussion about the meaning of Q and Q^*, see the chapter on *The Fourth Dimension*.

Take a matrix of the form $U = L_a R_b$ in $\mathrm{Mat}(4,\mathbb{R})$. Then $U^\top U = |a|^2|b|^2 I$, but in general $U + U^\top \neq \alpha I$. Take a matrix of the form $V = L_a + R_b$ in $\mathrm{Mat}(4,\mathbb{R})$. Then $V + V^\top = 2(\mathrm{Re}(a) + \mathrm{Re}(b))I$, but in general $V^\top V \neq \beta I$. Conversely, if $U \in \mathrm{Mat}(4,\mathbb{R})$ is such that $U + U^\top = \alpha I$ and $U^\top U = \beta I$ then the matrix U belongs either to $\mathbb{H}$ or to $\mathbb{H}^*$.

Besides real 4×4-matrices, quaternions can also be represented by complex 2×2-matrices:

$$w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \simeq \begin{pmatrix} w + iz & ix + y \\ ix - y & w - iz \end{pmatrix}.$$

The orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are represented by matrices obtained by multiplying each of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ by $i = \sqrt{-1}$:

$$\mathbf{i} \simeq \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{j} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

5.7 Linear spaces over $\mathbb{H}$

Much of the theory of linear spaces over commutative fields extends to $\mathbb{H}$. Because of the non-commutativity of $\mathbb{H}$ it is, however, necessary to distinguish between two types of linear spaces over $\mathbb{H}$, namely *right* linear spaces and *left* linear spaces.

A *right* linear space over $\mathbb{H}$ consists of an additive group V and a map

$$V \times \mathbb{H} \to V, \quad (\mathbf{x}, \lambda) \to \mathbf{x}\lambda$$

such that the usual distributivity and unity axioms hold and such that, for all λ, $\mu \in \mathbb{H}$ and $\mathbf{x} \in V$,

$$(\mathbf{x}\lambda)\mu = \mathbf{x}(\lambda\mu).$$

A *left* linear space over $\mathbb{H}$ consists of an additive group V and a map

$$\mathbb{H} \times V \to V, \ (\lambda, \mathbf{x}) \to \lambda\mathbf{x}$$

such that the usual distributivity and unity axioms hold and such that, for all λ, $\mu \in \mathbb{H}$ and $\mathbf{x} \in V$,

$$\lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}.$$

A mapping $L : V \to U$ between two right linear spaces V and U is a *right linear map* if it respects addition and, for all $\mathbf{x} \in V$, $\lambda \in \mathbb{H}$, $L(\mathbf{x}\lambda) = (L(\mathbf{x}))\lambda$.

Comment. In the matrix form the above definition means that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \lambda \right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1\lambda \\ x_2\lambda \end{pmatrix} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \lambda. \qquad \blacksquare$$

Remark. Although there are linear spaces over $\mathbb{H}$, there are no algebras over $\mathbb{H}$, since non-commutativity of $\mathbb{H}$ precludes bilinearity over $\mathbb{H}$: $\lambda(xy) = (\lambda x)y \neq (x\lambda)y = x(\lambda y) \neq x(y\lambda) = (xy)\lambda$. ∎

5.8 Function theory of quaternion variables

The richness of complex analysis suggests that there might be a function theory of quaternion variables. There are several different ways to generalize the theory of complex variables to the theory of quaternion functions of quaternion variables, $f : \mathbb{H} \to \mathbb{H}$. However, many generalizations are uninteresting, the classes of functions are too small or too large. In the following we will first eliminate the uninteresting generalizations.

First, consider quaternion differentiable functions such that

$$f'(q) = \lim_{h\to 0}[f(q+h) - f(q)]h^{-1}, \quad \text{where} \quad q, h \in \mathbb{H},$$

exists. The derivative $f'(q)$ is a real linear function

$$\mathbb{R}^4 \to \mathbb{R}^4 : h \to f'(q)h$$

corresponding to multiplication by a quaternion $a \in \mathbb{H}$ on the left, $f'(q)h = ah$ for $h \in \mathbb{H} = \mathbb{R}^4$. However, since $ah \neq ha$ the only quaternion differentiable functions are the affine right $\mathbb{H}$-linear functions

$$f(q) = aq + b \quad \text{where} \quad a, b \in \mathbb{H}.$$

We conclude that the set of quaternion differentiable functions reduces to a small and uninteresting set.

Second, if we consider power series in a quaternion variable $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, then we get the set of all power series in the four real variables w, x, y, z. For instance, the coordinates are first-order functions

$$\begin{aligned} w &= \tfrac{1}{4}(q - \mathbf{i}q\mathbf{i} - \mathbf{j}q\mathbf{j} - \mathbf{k}q\mathbf{k}), \\ x &= \tfrac{1}{4}(q - \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k})\mathbf{i}^{-1}, \\ y &= \tfrac{1}{4}(q + \mathbf{i}q\mathbf{i} - \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k})\mathbf{j}^{-1}, \\ z &= \tfrac{1}{4}(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} - \mathbf{k}q\mathbf{k})\mathbf{k}^{-1}, \end{aligned}$$

and so the set of power series in q, with left and right quaternion coefficients, is the set of all power series in the real variables w, x, y, z. This set is too big to be interesting.

Third, we could consider power series in q with real coefficients, that is, functions of type $f(q) = a_0 + a_1 q + a_2 q^2 + \ldots$ where $a_0, a_1, a_2, \ldots$ are real.

Restrict such a function to the complex subfield $\mathbb{C} \subset \mathbb{H}$, and send $z = x + iy$ to $f(z) = u+iv$, where $u = u(x, y)$ and $v = v(x, y)$. Decompose the quaternion q into real and vector parts, $q = q_0 + \mathbf{q}$, and note that $\mathbf{q}/|\mathbf{q}|$ is a generalized imaginary unit, $(\mathbf{q}/|\mathbf{q}|)^2 = -1$. Then

$$f(q_0 + \mathbf{q}) = u(q_0, |\mathbf{q}|) + \frac{\mathbf{q}}{|\mathbf{q}|}v(q_0, |\mathbf{q}|).$$

So this generalization just rotates the graph of $\mathbb{C} \to \mathbb{C}$, $z \to f(z)$, or rather makes $i = \mathbf{i}$ sweep all of $S^2 = \{\mathbf{r} \in \mathbb{R}^3 \mid |\mathbf{r}| = 1\}$, and thus gives only (a subclass of) axially symmetric functions.

Fourth, we could consider functions which are conformal almost everywhere in $\mathbb{R}^4$. This leads to Möbius transformations of $\mathbb{R}^4$, or its one-point compactification $\mathbb{R}^4 \cup \{\infty\}$. The Möbius transformations are compositions of the four mappings sending q to

aqb^{-1}	$a, b \in S^3$	rotations
$q + b$	$b \in \mathbb{H}$	translations
$q\lambda$	$\lambda > 0$	dilations
$(q^{-1} + c)^{-1}$	$c \in \mathbb{H}$	transversions.

A nice thing about quaternions is that all Möbius transformations of $\mathbb{R}^4$ can be written in the form $(aq + b)(cq + d)^{-1}$, where $a, b, c, d \in \mathbb{H}$.

Fifth, we could focus our attention on a generalization of the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial w} + \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = 0 \quad \text{where} \quad f : \mathbb{H} \to \mathbb{H}.$$

Using the differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

the above equation can be put into the form

$$\frac{\partial f_0}{\partial w} + \frac{\partial \mathbf{f}}{\partial w} + \nabla f_0 - \nabla \cdot \mathbf{f} + \nabla \times \mathbf{f} = 0$$

where $f = f_0 + \mathbf{f}$ with $f_0 : \mathbb{H} \to \mathbb{R}$ and $\mathbf{f} : \mathbb{H} \to \mathbb{R}^3$. This decomposes into scalar and vector parts

$$\frac{\partial f_0}{\partial w} - \nabla \cdot \mathbf{f} = 0 \quad \text{and} \quad \frac{\partial \mathbf{f}}{\partial w} + \nabla f_0 + \nabla \times \mathbf{f} = 0.$$

There are three linearly independent first-order solutions to these equations

$$q_x = x - \mathbf{i}w, \; q_y = y - \mathbf{j}w, \; q_z = z - \mathbf{k}w.$$

Higher-order homogeneous solutions are linear combinations of symmetrized products of q_x, q_y, q_z. For instance, the symmetrized product of degrees $2, 1, 0$ with respect to q_x, q_y, q_z is seen to be

$$q_x^2 q_y + q_x q_y q_x + q_y q_x^2 = 3(x^2 - w^2)y - 6wxy\mathbf{i} + (w^3 - 3wx^2)\mathbf{j}.$$

This already shows that the last alternative results in an interesting class of new functions, to some extent analogous to the class of holomorphic functions of a complex variable.

Historical survey

Hamilton invented his quaternions in 1843 when he tried to introduce a product for vectors in $\mathbb{R}^3$ similar to the product of complex numbers in $\mathbb{C}$. The present-day formalism of vector algebra was extracted out of the quaternion product of two vectors, $\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$, by Gibbs in 1901.

Hamilton tried to find an algebraic system which would do for the space $\mathbb{R}^3$ the same thing as complex numbers do for the plane $\mathbb{R}^2$. In particular, Hamilton wanted to find a multiplication rule for triplets $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ so that $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$, that is, a multiplicative product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. However, no such bilinear products exist (at least not over the rationals), since $3 \times 21 = 63 \neq n_1^2 + n_2^2 + n_3^2$ for any integers n_1, n_2, n_3 though $3 = 1^2 + 1^2 + 1^2$ and $21 = 1^2 + 2^2 + 4^2$ (no integer of the form $4^a(8b+7)$, with $a \geq 0$, $b \geq 0$, is a sum of three squares, a result of Legendre in 1830).

Hamilton also tried to find a generalized complex number system in three dimensions. However, no such associative hypercomplex numbers exist in three dimensions. This can be seen by considering generalized imaginary units $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = -1$, and such that $1, \mathbf{i}, \mathbf{j}$ span $\mathbb{R}^3$. [3] The product must be of the form $\mathbf{ij} = \alpha + \mathbf{i}\beta + \mathbf{j}\gamma$ for some real α, β, γ. Then

$$\begin{aligned} \mathbf{i}(\mathbf{ij}) &= \mathbf{i}\alpha - \beta + (\mathbf{ij})\gamma = \mathbf{i}\alpha - \beta + (\alpha + \mathbf{i}\beta + \mathbf{j}\gamma)\gamma \\ &= -\beta + \alpha\gamma + \mathbf{i}(\alpha + \beta\gamma) + \mathbf{j}\gamma^2, \end{aligned}$$

whereas by associativity $\mathbf{i}(\mathbf{ij}) = \mathbf{i}^2\mathbf{j} = -\mathbf{j}$ which leads to a contradiction since $\gamma^2 \geq 0$ for all real γ.

Hamilton's great idea was to go to four dimensions and consider elements of the form $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ where the hypercomplex units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the following **non-commutative** multiplication rules

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \\ &\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \end{aligned}$$

3 Actually, it is not necessary to assume that $\mathbf{j}^2 = -1$. The computation shows that there is no embedding $\mathbb{C} \subset \mathbb{R}^3$, where $\mathbb{R}^3$ is an associative algebra.

Hamilton named his four-component elements **quaternions**. Quaternions form a division ring which we have denoted by $\mathbb{H}$ in honor of Hamilton.

Cayley in 1845 was the first one to publish the quaternionic representation of rotations of $\mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{r} \to a\mathbf{r}a^{-1}$, but he mentioned that the result was known to Hamilton. Cayley, in 1855, also discovered the quaternionic representation of 4-dimensional rotations:

$$\mathbb{R}^4 \to \mathbb{R}^4, \quad q \to aqb^{-1},$$

where we have identified $\mathbb{R}^4 = \mathbb{H}$.

The differential operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is due to Hamilton, although his symbol for nabla was turned 30°. The first one to study solutions of

$$\frac{\partial f}{\partial w} + \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = 0, \quad \text{where} \quad f : \mathbb{H} \to \mathbb{H},$$

was Fueter 1935.

Comment

The quaternion formalism might seem awkward to a physicist or an engineer, for two reasons: first, the squares of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are negative, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and second, one invokes a 4-dimensional space which is beyond our ability of visualization.

Exercises

1. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^3$, $|\mathbf{u}| = 1$. Show that $\mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{x} \to \mathbf{uxu}$ is a reflection across the plane $\mathbf{u}^\perp$.
2. Determine square roots of the quaternion $q = q_0 + \mathbf{q}$.
3. Hurwitz integral quaternions $q = w + ix + jy + kz$ are $\mathbb{Z}$-linear combinations of i, j, k and $\frac{1}{2}(1 + i + j + k)$, that is, either all w, x, y, z are integers or of the form $n + \frac{1}{2}$. Show that $|q|^2$ is an integer, and that the set

$$\{w + ix + jy + kx \mid w, x, y, z \in \mathbb{Z} \quad \text{or} \quad w, x, y, z \in \mathbb{Z} + \tfrac{1}{2}\}$$

is closed under multiplication.

4. Clearly, $ab = ba$ implies $e^a e^b = e^{a+b}$, but does $e^a e^b = e^{a+b}$ imply $ab = ba$?
5. Denote

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^- = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} a\bar{d}d - b\bar{d}c & c\bar{b}b - d\bar{b}a \\ b\bar{c}c - a\bar{c}d & d\bar{a}a - c\bar{a}b \end{pmatrix}^{-}$$

for a non-zero $\Delta = |a|^2|d|^2 + |b|^2|c|^2 - 2\,\mathrm{Re}(a\,\bar{c}\,d\,\bar{b})$.

6. Verify that only one of the matrices

$$a = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$$

is invertible.

7. Does an involutory automorphism of the real algebra $\mathrm{Mat}(2, \mathbb{H})$ necessarily send a diagonal matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{where} \quad a \in \mathbb{H}$$

to a diagonal matrix?

8. Suppose A $(\neq \mathbb{R})$ is a simple real associative algebra of dimension ≤ 4 with center $\mathbb{R}$. Show that A is $\mathbb{H}$ or $\mathrm{Mat}(2, \mathbb{R})$.
9. Suppose A $(\neq \mathbb{R})$ is a simple real associative algebra with center $\mathbb{R}$ and an anti-automorphism $x \to \alpha(x)$ such that $x + \alpha(x) \in \mathbb{R}$ and $x\alpha(x) \in \mathbb{R}$. Show that A is $\mathbb{H}$ or $\mathrm{Mat}(2, \mathbb{R})$.
10. Show that all the subgroups of $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ are normal, that is, for a subgroup $H \subset Q_8$ and elements $g \in Q_8$, $h \in H$, $ghg^{-1} \in H$.
11. Take two vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$, such that $|\mathbf{a}| = |\mathbf{b}|$, and $a = e^{\mathbf{a}}$, $b = e^{\mathbf{b}}$ in S^3. Determine the point-wise invariant plane of the simple rotation $q \to aqb^{-1}$ of $\mathbb{R}^4$.

Solutions

2. If $q = 0$, then there is only one square root, 0. If $\mathbf{q} = 0$, $q_0 > 0$, then there are two square roots, $\pm\sqrt{q_0}$. If $\mathbf{q} = 0$, $q_0 < 0$, then there is an infinity of square roots, $\sqrt{-q_0}\,\mathbf{u}$, where $\mathbf{u}$ is a unit pure quaternion $\mathbf{u} \in \mathbb{R}^3 \subset \mathbb{H}$, $|\mathbf{u}| = 1$. If $\mathbf{q} \neq 0$, then there are two square roots,

$$\sqrt{\tfrac{1}{2}(|q| + q_0)} + \frac{\mathbf{q}}{|\mathbf{q}|}\sqrt{\tfrac{1}{2}(|q| - q_0)}$$

and its opposite.

4. Hint: consider the quaternions $a = 3\pi i$ and $b = 4\pi j$, or the matrices

$$a = 3\pi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = 4\pi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

6. a is invertible, but b is not.
11. If $a = b^{-1}$, the point-wise invariant plane is $\mathbf{a}^\perp$ in $\mathbb{R}^3$. Otherwise the point-wise invariant plane is spanned by $\mathbf{a} + \mathbf{b}$ and $|\mathbf{a}||\mathbf{b}| - \mathbf{ab} = |\mathbf{a}||\mathbf{b}| + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \times \mathbf{b}$.

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6

The Fourth Dimension

In this chapter we study the geometry of the Euclidean space $\mathbb{R}^4$. The purpose is to help readers to get a solid view, or as solid a view as possible, of the first dimension beyond our ability to visualize. This is an important intermediate step in scrutinizing higher dimensions. We start by reviewing regular figures in lower dimensions.

6.1 Regular polygons in $\mathbb{R}^2$

The equilateral triangle, the square, the regular pentagon, ..., are regular polygons. We shall also call them a 3-cell, 4-cell, 5-cell, ..., denoted by $\{3\}$, $\{4\}$, $\{5\}$, ..., respectively. Therefore, we call a regular p-gon a p-cell, denoted by $\{p\}$. As p grows toward infinity, we get in the limit an ∞-cell, where the line is divided into line segments of equal length. As a degenerate case we get a 2-cell, which is bounded by 2 line segments in the same place. The interior angle of a regular p-gon at a vertex is $(1-2/p)\pi$.

6.2 Regular polyhedra in $\mathbb{R}^3$

A regular polyhedron is a convex polyhedron bounded by congruent regular polygons, for instance, by p-gons. The number of regular p-gons meeting at a vertex is the same, say q; it satisfies

$$q\left(1-\frac{2}{p}\right)\pi < 2\pi,$$

because the sum of angles of faces meeting at a vertex cannot exceed 2π. The above inequality can also be written in the form

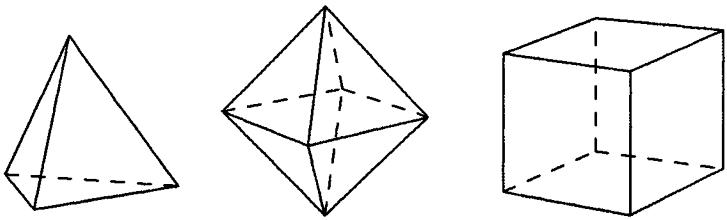
$$\frac{1}{p}+\frac{1}{q}>\frac{1}{2}.$$

The same result is obtained by inspection of the topological properties of a regular polyhedron: the numbers N_0, N_1, N_2 of vertices, edges and faces satisfy the Euler formula:

$$N_0 - N_1 + N_2 = 2.$$

On the other hand, each edge of a regular polyhedron is a boundary of two faces, each with p sides, so that $2N_1 = pN_2$; and a vertex is a meeting point of q edges, each with 2 end points, so that $qN_0 = 2N_1$. The above inequality is a consequence of the Euler formula and the equation

$$qN_0 = 2N_1 = pN_2.$$



3,3 3,4 4,3

A regular polyhedron ($p, q \geq 3$) must satisfy the foregoing inequality, and so only a few pairs p, q are possible. These regular polyhedra are called Platonic solids, or p, q-cells with *Schläfli* symbols $\{p, q\}$. There are five Platonic solids.

Name	$\{p,q\}$	N_0	N_1	N_2
Tetrahedron	$\{3,3\}$	4	6	4
Octahedron	$\{3,4\}$	6	12	8
Cube	$\{4,3\}$	8	12	6
Icosahedron	$\{3,5\}$	12	30	20
Dodecahedron	$\{5,3\}$	20	30	12

When $q = 2$ in the above inequality we get a dihedron with Schläfli symbol $\{p, 2\}$. A dihedron is bounded by two regular polygons positioned in the same place.

When a plane is covered by regular polygons so that at each vertex there meet q regular p-gons, we are solving the equation

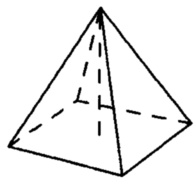
$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

There are three solutions to the above equation; they have Schläfli symbols

$\{4,4\}$, $\{3,6\}$ and $\{6,3\}$ corresponding to tilings of the plane by squares, equilateral triangles and regular hexagons. These regular tilings are called *tessellations*.

6.3 Regular polytopes in $\mathbb{R}^4$

A polyhedron is regular if its faces and vertices (= parts of the polyhedron near a vertex point) are regular. A regular polyhedron with Schläfli symbol $\{p,q\}$ has p-cells as faces and q-cells as vertices. A vertex is regular, if a plane cuts off a regular polygon whose central normal passes through the vertex.



A regular vertex

A polytope is a higher-dimensional analog of a polyhedron. A polytope is regular if its faces and vertices are regular. A 4-dimensional regular polytope with p,q-cells as faces must have q,r-cells as vertices. This drops the number of 4-dimensional regular polytopes from $5^2 = 25$ to 11. The sum of the solid angles of the faces meeting at a vertex cannot exceed 4π. As a consequence, there remain six possible combinations of p,q and q,r. A closer inspection shows that all these six combinations are in fact 4-dimensional regular polytopes; we shall call them p,q,r-cells with Schläfli symbols $\{p,q,r\}$.

$\{p,q,r\}$	N_0	N_1	N_2	N_3	Face	Vertex
$\{3,3,3\}$	5	10	10	5	Tetrahedron	Tetrahedron
$\{3,3,4\}$	8	24	32	16	Tetrahedron	Octahedron
$\{4,3,3\}$	16	32	24	8	Cube	Tetrahedron
$\{3,4,3\}$	24	96	96	24	Octahedron	Cube
$\{3,3,5\}$	120	720	1200	600	Tetrahedron	Icosahedron
$\{5,3,3\}$	600	1200	720	120	Dodecahedron	Tetrahedron

There are the regular simplex $\{3,3,3\}$ and the hypercube $\{4,3,3\}$, also called a tesseract. There is the octahedron analog $\{3,3,4\}$, a dipyramid with octahedron as a basis. There are the analogs of the icosahedron and the dodecahedron, $\{3,3,5\}$ and $\{5,3,3\}$; and there is an extra regular polytope $\{3,4,3\}$.

The 3-dimensional space can be filled with cubes, a configuration with

Schläfli symbol $\{4,3,4\}$. The 4-dimensional space can be filled with hypercubes, dipyramids and the extra regular polytope, configurations with Schläfli symbols $\{4,3,3,4\}$, $\{3,3,4,3\}$ and $\{3,4,3,3\}$.

In a higher-dimensional space, $n > 4$, there are only the regular simplex, dipyramid and hypercube, and it can only be filled with hypercubes.

6.4 The spheres

A circle with radius r in $\mathbb{R}^2$ has circumference $2\pi r$ and area πr^2. A sphere with radius r in $\mathbb{R}^3$ has surface $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$. A hypersphere with radius r in $\mathbb{R}^4$ has 3-dimensional surface $2\pi^2 r^3$ and 4-dimensional hypervolume $\frac{1}{2}\pi^2 r^4$. For lower-dimensional spheres we have the following table:

n	surface	volume
1	2	$2r$
2	$2\pi r$	πr^2
3	$4\pi r^2$	$\frac{4}{3}\pi r^3$
4	$2\pi^2 r^3$	$\frac{1}{2}\pi^2 r^4$
5	$\frac{8}{3}\pi^2 r^4$	$\frac{8}{15}\pi^2 r^5$

If the volume of the sphere in $\mathbb{R}^n$ is denoted by $\omega_n r^n$ then its surface is $n\omega_n r^{n-1}$. Observe a rule $m\omega_m r^{m-1} = 2\pi r \cdot \omega_n r^n$ between the surface in dimension $m = n+2$ and the volume in dimension n. This leads to the recursion

$$\omega_{n+2} = \frac{2\pi\omega_n}{n+2}$$

and the formula

$$\omega_n = \frac{\pi^{n/2}}{(n/2)!}$$

which can be computed for odd n by recalling that $(1/2)! = \sqrt{\pi}/2$.

6.5 Rotations in four dimensions

Let A be an antisymmetric 4×4-matrix, that is, $A \in \mathrm{Mat}(4,\mathbb{R})$, $A^\top = -A$. Then the matrix e^A represents a rotation of the 4-dimensional Euclidean space $\mathbb{R}^4$. In general, a rotation of $\mathbb{R}^4$ has two invariant planes which are completely orthogonal; in particular they have only one point in common. The antisymmetric matrix A has imaginary eigenvalues, say $\pm i\alpha$ and $\pm i\beta$, the eigenvalues of the rotation matrix e^A are unit complex numbers $e^{\pm i\alpha}$ and $e^{\pm i\beta}$, and the invariant planes turn by angles α and β under e^A. First, assume that $\alpha > \beta \geq 0$ (and $\alpha < \pi$). Each vector is turned through at least an angle β and at most an angle α. In the case $\beta = 0$ we have a simple rotation leaving one plane point-wise fixed. If β/α is rational, then $e^{tA} = I$ for some $t > 0$. If β/α is irrational, then $e^{tA} \neq I$ for any $t > 0$.

By the Cayley-Hamilton theorem e^A is a linear combination of the matrices I, A, A^2 and A^3 so that

$$e^A = h_0 I + h_1 A + h_2 A^2 + h_3 A^3$$

and direct computation shows that

$$\begin{aligned}
h_0 &= \frac{1}{\alpha^2 - \beta^2}(\alpha^2 \cos\beta - \beta^2 \cos\alpha), \\
h_1 &= \frac{1}{\alpha^2 - \beta^2}\left(\frac{\alpha^2}{\beta}\sin\beta - \frac{\beta^2}{\alpha}\sin\alpha\right), \\
h_2 &= \frac{1}{\alpha^2 - \beta^2}(\cos\beta - \cos\alpha), \\
h_3 &= \frac{1}{\alpha^2 - \beta^2}\left(\frac{1}{\beta}\sin\beta - \frac{1}{\alpha}\sin\alpha\right).
\end{aligned}$$

Letting α now approach β and computing the coefficients in the limit give

$$\begin{aligned}
\lim_{\alpha\to\beta} e^A &= I(\cos\alpha + \frac{\alpha}{2}\sin\alpha) \\
&\quad + \frac{A}{\alpha}\left(\frac{3}{2}\sin\alpha - \frac{\alpha}{2}\cos\alpha\right) \\
&\quad + \frac{A^2}{\alpha^2}\left(\frac{\alpha}{2}\sin\alpha\right) \\
&\quad + \frac{A^3}{\alpha^3}\left(\frac{1}{2}\sin\alpha - \frac{\alpha}{2}\cos\alpha\right).
\end{aligned}$$

Observe that in the limit $A^2 = -\alpha^2 I$, which cancels some terms and results in

$$\lim_{\alpha\to\beta} e^A = I\cos\alpha + \frac{A}{\alpha}\sin\alpha.$$

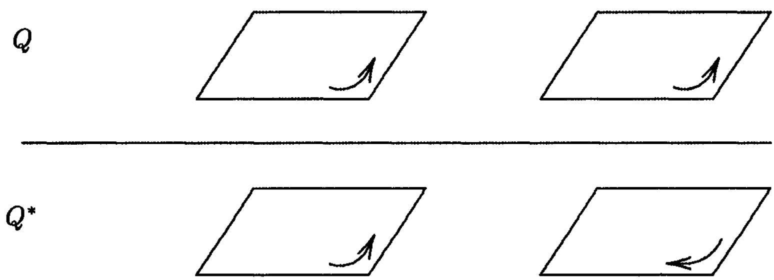
These rotations with only one rotation angle α have a whole bundle of invariant rotation planes. In fact, every point of $\mathbb{R}^4$ stays in some invariant plane, but not every plane of $\mathbb{R}^4$ is an invariant plane of e^A.

If a rotation U of $\mathbb{R}^4$ has rotation angles α and β we shall denote it by $U(\alpha, \beta)$. Consider the set $\mathcal{J} = \{U(\alpha,\beta) \in SO(4) \mid \alpha = \beta\}$ and the relation '$\sim$' in the set $\mathcal{J}' = \mathcal{J} \setminus \{I, -I\}$,

$$U \sim V \Longleftrightarrow UV \in \mathcal{J},$$

which can be seen to be an equivalence relation. The equivalence class of a matrix $U \in \mathcal{J}'$ is the set $\{X \in \mathcal{J}' \mid X \sim U\}$. This equivalence class together with the center $\{I, -I\}$ of the rotation group $SO(4)$ forms a subgroup of $SO(4)$, denoted in the sequel by the letter Q. Also $(\mathcal{J} \setminus Q) \cup \{I, -I\}$ is a subgroup of $SO(4)$; denote it by Q^*. Observe that $UV = VU$ for $U \in Q$ and $V \in Q^*$. It can be shown that Q and Q^* are isomorphic to the group of unit quaternions $S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$.

Each rotation $L \in SO(4)$ of $\mathbb{R}^4$ can be written in the form $L = UV$, where $U \in Q$, $V \in Q^*$. The rotation angles of L are $\alpha \pm \beta$ when the rotation angles of U and V are α and β. A pair of completely orthogonal planes, both with a fixed sense of rotation, induces a pair of senses of rotations for all pairs of completely orthogonal planes. There are two classes of such pairs of oriented planes: those of the type Q and those of type Q^*.



Furthermore, we have an isomorphism of algebras,

$$\mathbb{H} \simeq \{\lambda q \mid \lambda > 0,\ q \in Q\} \cup \{0\},$$

which we shall regard as an identification. Introduce the algebra

$$\mathbb{H}^* = \{\lambda q \mid \lambda > 0,\ q \in Q^*\} \cup \{0\}.$$

and observe an isomorphism of algebras, $\mathbb{H} \simeq \mathbb{H}^*$.

6.6 Rotating ball in $\mathbb{R}^4$

A rotating ball in $\mathbb{R}^3$ has an axis of rotation, like the axis going through the North and South Poles, and a plane of rotation, like the plane of the equator. A rotating ball in $\mathbb{R}^4$ has two planes of rotation, which are completely orthogonal to each other in the sense that they have only one point in common. Let the angular velocities in these planes be bivectors ω_1 and ω_2. The total angular velocity is a bivector $\omega = \omega_1 + \omega_2$. The velocity $\mathbf{v}$ of a point $\mathbf{x}$ on the surface of the ball is

$$\mathbf{v} = \mathbf{x} \,\lrcorner\, \omega_1 + \mathbf{x} \,\lrcorner\, \omega_2.$$

Assume that φ is the angle between the direction $\mathbf{x}$ and the plane of ω_1. Then

$$|\mathbf{v}| = |\mathbf{x}|\sqrt{|\omega_1|^2(\cos\varphi)^2 + |\omega_2|^2(\sin\varphi)^2}.$$

Therefore, the local angular velocity $|\mathbf{v}|/|\mathbf{x}|$ is always between $|\omega_1|$ and $|\omega_2|$.

If $|\omega_1| = |\omega_2|$, then every point on the sphere is rotating at the same velocity and furthermore every point is travelling along some great circle, that is, everybody is living on an equator!

6.7 The Clifford algebra $\mathcal{C}\ell_4$

The Clifford algebra $\mathcal{C}\ell_4$ of $\mathbb{R}^4$ with an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is generated by the relations

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_4^2 = 1 \quad \text{and} \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j.$$

It is a 16-dimensional algebra with basis consisting of

scalar	1
vectors	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
bivectors	$\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$
3-vectors	$\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}$
volume element	$\mathbf{e}_{1234}$

where $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j$ for $i \neq j$ and $\mathbf{e}_{1234} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$.

An arbitrary element $u \in \mathcal{C}\ell_4$ is a sum of its k-vector parts:

$$u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4 \quad \text{where} \quad \langle u\rangle_k \in \bigwedge^k \mathbb{R}^4.$$

There are three important involutions of $\mathcal{C}\ell_4$:

$$\begin{aligned}
\hat{u} &= \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4 && \text{grade involution}\\
\tilde{u} &= \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4 && \text{reversion}\\
\bar{u} &= \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4 && \text{Clifford-conjugation.}
\end{aligned}$$

The Clifford algebra $\mathcal{C}\ell_4$ is isomorphic to the real algebra of 2×2-matrices $\mathrm{Mat}(2, \mathbb{H})$ with quaternions as entries,

$$\mathbf{e}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

6.8 Bivectors in $\bigwedge^2 \mathbb{R}^4 \subset \mathcal{C}\ell_4$

The essential difference between 3-dimensional and 4-dimensional spaces is that bivectors are no longer products of two vectors. Instead, bivectors are sums of products of two vectors in $\mathbb{R}^4$. In the 3-dimensional space $\mathbb{R}^3$ there are only *simple* bivectors, that is, all the bivectors represent a plane. In the 4-dimensional space $\mathbb{R}^4$ this is not the case any more.

Example. The bivector $\mathbf{B} = \mathbf{e}_{12} + \mathbf{e}_{34} \in \bigwedge^2 \mathbb{R}^4$ is not simple. For all simple elements the square is real, but $\mathbf{B}^2 = -2 + 2\mathbf{e}_{1234} \notin \mathbb{R}$. ∎

If the square of a bivector is real, then it is simple. [1]

Usually a bivector in $\bigwedge^2 \mathbb{R}^4$ can be uniquely written as a sum of two simple bivectors, which represent completely orthogonal planes. There is an exception to this uniqueness, crucial to the study of four dimensions: If the simple components of a bivector have equal squares, that is equal norms, then the decomposition to a sum of simple components is not unique.

Example. The bivector $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4$ can also be decomposed into a sum of two completely orthogonal bivectors as follows:

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3)(\mathbf{e}_2 + \mathbf{e}_4) + \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_4).$$ ∎

6.9 The group Spin(4) and its Lie algebra

The group $\mathbf{Spin}(4) = \{s \in \mathcal{C}\ell_4^+ \mid s\tilde{s} = 1\}$ is a two-fold covering group of the rotation group $SO(4)$ so that the map

$$\mathbb{R}^4 \to \mathbb{R}^4, \ \mathbf{x} \to s\mathbf{x}s^{-1}, \quad \text{where} \quad s \in \mathbf{Spin}(4),$$

is a rotation, and each rotation can be so represented, the same rotation being obtained by s and $-s$. The Lie algebra of **Spin**(4) is the subspace of bivectors $\bigwedge^2 \mathbb{R}^4$ with commutator product as the product. The two sets of basis bivectors

$$\begin{array}{lcl} \frac{1}{4}(\mathbf{e}_{23} + \mathbf{e}_{14}) & & \frac{1}{4}(\mathbf{e}_{23} - \mathbf{e}_{14}) \\ \frac{1}{4}(\mathbf{e}_{31} + \mathbf{e}_{24}) & \text{and} & \frac{1}{4}(\mathbf{e}_{31} - \mathbf{e}_{24}) \\ \frac{1}{4}(\mathbf{e}_{12} + \mathbf{e}_{34}) & & \frac{1}{4}(\mathbf{e}_{12} - \mathbf{e}_{34}) \end{array}$$

in $\bigwedge^2 \mathbb{R}^4 \subset \mathcal{C}\ell_4$ both span a Lie algebra isomorphic to the subspace $\bigwedge^2 \mathbb{R}^3 \subset \mathcal{C}\ell_3$ with basis $\{\frac{1}{2}\mathbf{e}_{23}, \frac{1}{2}\mathbf{e}_{31}, \frac{1}{2}\mathbf{e}_{12}\}$, that is, they satisfy the same commutation relations. In other words, the Lie algebras

$$\frac{1}{2}(1 - \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4 \quad \text{and} \quad \frac{1}{2}(1 + \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4$$

1 Although the square of a 3-vector is real, it need not be simple. For instance, $\mathbf{V} = \mathbf{e}_{123} + \mathbf{e}_{456} \in \bigwedge^3 \mathbb{R}^6$ is not simple [this can be seen by computing $\mathbf{V}\mathbf{e}_i\mathbf{V}^{-1}$, $i = 1, 2, \ldots, 6$, and observing that they are not all vectors].

are both isomorphic to $\bigwedge^2 \mathbb{R}^3$. The two subspaces $\frac{1}{2}(1 \pm \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4$ of $C\ell_4$ annihilate each other, and consequently,

$$\bigwedge^2 \mathbb{R}^4 \simeq \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3.$$

At the group level this means the isomorphism

$$\mathbf{Spin}(4) \simeq \mathbf{Spin}(3) \times \mathbf{Spin}(3)$$

where $\mathbf{Spin}(3) \simeq S^3 \simeq SU(2)$.

6.10 The mapping $\mathbf{F} \to (1+\mathbf{F})(1-\mathbf{F})^{-1}$ for $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$

The exponential $e^{\mathbf{F}/2} \in \mathbf{Spin}(4)$ of a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ corresponds to the rotation $e^A \in SO(4)$, where $A(\mathbf{x}) = \mathbf{F} \llcorner \mathbf{x}$, for $\mathbf{x} \in \mathbb{R}^4$. Every rotation of $\mathbb{R}^4$ can be so represented, and the two elements $\pm e^{\mathbf{F}/2}$ represent the same rotation.

The exterior exponential $e^{\mathbf{F}} = 1 + \mathbf{F} + \frac{1}{2}\mathbf{F} \wedge \mathbf{F}$ of a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is a multiple of an element in $\mathbf{Spin}(4)$, that is,

$$\frac{e^{\mathbf{F}}}{|e^{\mathbf{F}}|} \in \mathbf{Spin}(4).$$

Up to a sign, every element in $\mathbf{Spin}(4)$ can be so represented, except $\pm\mathbf{e}_{1234}$. The exterior exponential $e^{\mathbf{F}}$ of the bivector $\mathbf{F}$ corresponds to the rotation $(I+A)(I-A)^{-1}$; every rotation of $\mathbb{R}^4$ can be so represented, except $-I$.

The above observations raise the question: What is the rotation corresponding to $(1+\mathbf{F})(1-\mathbf{F})^{-1} \in \mathbf{Spin}(4)$? This is an interesting and non-trivial question in dimension 4. [2] Here follows the answer.

Let $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$. The antisymmetric function induced by $\mathbf{F}$ is denoted by A, that is, $A(\mathbf{x}) = \mathbf{F} \llcorner \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^4$. Write $s = (1+\mathbf{F})(1-\mathbf{F})^{-1}$. The rotation induced by $s \in \mathbf{Spin}(4)$ is denoted by $U \in SO(4)$, that is, $U = (I+A)(I-A)^{-1}$. In other words, $U(\mathbf{x}) = s\mathbf{x}s^{-1}$ for all $\mathbf{x} \in \mathbb{R}^4$. The following cases can be distinguished:

(i) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^3$ then $U = \left(\dfrac{I+A}{I-A}\right)^2$.

(ii) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is simple, then $U = \left(\dfrac{I+A}{I-A}\right)^2$.

(iii) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is isoclinic, then $U = \dfrac{I+2A}{I-2A}$.

2 It is also a non-trivial question in dimension 5. In dimension 6, $(1+\mathbf{F})(1-\mathbf{F})^{-1} \notin \mathbf{Spin}(6)$.

(iv) In the case of an arbitrary $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ we cannot express U as a rational function of A [although U still has the same eigenplanes as A]. Instead,

$$U = \frac{A^4 + B^4 - 2A^2B^2 + 6A^2 - 2B^2 + I + 4A(A^2 - B^2 + I)}{A^4 + B^4 - 2A^2B^2 - 2A^2 - 2B^2 + I},$$

where $B(\mathbf{x}) = (\mathbf{F}\mathbf{e}_{1234}) \,\llcorner\, \mathbf{x}$, the dual of A. The denominator of U is a multiple of the identity I. [3]

Summary

There are three different kinds of rotations in four dimensions depending on the values of the rotation angles α, β satisfying $\pi > \alpha \geq \beta \geq 0$. Let $R : \mathbb{R}^4 \to \mathbb{R}^4$ be a rotation and $\mathbf{a}$ a non-zero vector with iterated images $\mathbf{b} = R(\mathbf{a})$, $\mathbf{c} = R(\mathbf{b})$, $\mathbf{d} = R(\mathbf{c})$. In general, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \neq 0$. In the case of a simple rotation with $\beta = 0$, only the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \neq 0$ but $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} = 0$. In the case of an isoclinic [4] rotation with $\alpha = \beta$, only the vectors $\mathbf{a}, \mathbf{b}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \neq 0$ but $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$ and $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{d} = 0$.

In general, a rotation of $\mathbb{R}^4$ has six parameters, computed as

$$(3 + 2 - 1) + 2 = 6.$$

The number 3 comes from picking up a unit vector $\mathbf{a}$; the number 2 comes from picking up a unit vector $\mathbf{b}$ in the orthogonal complement of $\mathbf{a}$; the unit bivector $\mathbf{ab} = \mathbf{a} \wedge \mathbf{b}$ fixes a plane but the same plane is obtained by rotating $\mathbf{a}$ and $\mathbf{b}$ in the plane of $\mathbf{a} \wedge \mathbf{b}$, thus subtract 1; then finally add 2 for the two rotation parameters/angles α and β. On the other hand, an isoclinic rotation has three parameters, computed as

$$(3 - 1) + 1 = 3.$$

The number 3 comes from picking up a unit vector $\mathbf{a}$ in S^3; but in an isoclinic rotation $\mathbf{a}$ stays in a plane or a great circle S^1, so subtract 1; and finally add 1 for the rotation/angle $\alpha = \beta$.

A simple bivector, an exterior product of two vectors, corresponds to simple

3 In dimension 5 the rotation U is given by the same expression, when

$$B(\mathbf{x}) = \Big(\mathbf{F}\frac{\mathbf{F} \wedge \mathbf{F}}{|\mathbf{F} \wedge \mathbf{F}|}\Big) \,\llcorner\, \mathbf{x}.$$

The denominator is no longer a multiple of I, although it still commutes with the numerator by virtue of $AB = BA$.

4 An isoclinic rotation with equal rotation angles corresponds to a multiplication by a quaternion.

rotation turning only one plane. A simple bivector multiplied by one of the idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{1234})$ corresponds to an isoclinic rotation. An isoclinic rotation has an infinity of rotation planes, and in fact, each vector is in some invariant rotation plane of an isoclinic rotation.

The two-fold cover $\mathbf{Spin}(4)$ of $SO(4)$ has three different subgroups isomorphic to $\mathbf{Spin}(3)$, each with a Lie algebra

$$\bigwedge^2 \mathbb{R}^3, \quad \frac{1}{2}(1+\mathbf{e}_{1234})\bigwedge^2 \mathbb{R}^4, \quad \frac{1}{2}(1-\mathbf{e}_{1234})\bigwedge^2 \mathbb{R}^4.$$

There is an automorphism of $\mathbf{Spin}(4)$ which swaps the last two copies of $\mathbf{Spin}(3)$, but there is no automorphism of $\mathbf{Spin}(4)$ swapping the first copy of $\mathbf{Spin}(3)$ with either of the other two copies.

Exercises

1. Compute the squares of $\frac{1}{2}(1+\mathbf{e}_{12}+\mathbf{e}_{34} \pm \mathbf{e}_{1234})$.
2. Take a vector $\mathbf{a} \in \mathbb{R}^4$ and a bivector $\mathbf{B} = \alpha\mathbf{e}_{12} + \beta\mathbf{e}_{34} \in \bigwedge^2 \mathbb{R}^4$. Show that $\mathbf{BaB} \in \mathbb{R}^4$.
3. Compute $\exp(\alpha\mathbf{e}_{12} + \beta\mathbf{e}_{34})$.
4. Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. Compute $\mathbf{A} = \mathbf{a}\mathbf{e}_{123}$ and $\mathbf{B} = \mathbf{b}\mathbf{e}_{123}$. Determine $\frac{1}{2}(1+\mathbf{e}_{1234})\mathbf{A}$ and $\frac{1}{2}(1-\mathbf{e}_{1234})\mathbf{B}$, and show that these bivectors commute.
5. Compute $\mathbf{C} = \frac{1}{2}(1+\mathbf{e}_{1234})\mathbf{A} + \frac{1}{2}(1-\mathbf{e}_{1234})\mathbf{B}$, and express $\exp(\mathbf{C})$ using $|\mathbf{a}|$ and $|\mathbf{b}|$. What are the two rotation angles of the rotation $\mathbb{R}^4 \to \mathbb{R}^4$, $\mathbf{x} \to c\mathbf{x}c^{-1}$ where $c = \exp(\mathbf{C})$?
6. Consider the Lie algebra $\bigwedge^2 \mathbb{R}^4$ with the commutator product $[a,b] = ab - ba$, and its three subalgebras spanned by

$$\begin{aligned} \mathcal{V}: &\quad \tfrac{1}{2}\mathbf{e}_{23},\ \tfrac{1}{2}\mathbf{e}_{31},\ \tfrac{1}{2}\mathbf{e}_{12} \\ \mathcal{I}_1: &\quad \tfrac{1}{4}(\mathbf{e}_{23}-\mathbf{e}_{14}),\ \tfrac{1}{4}(\mathbf{e}_{31}-\mathbf{e}_{24}),\ \tfrac{1}{4}(\mathbf{e}_{12}-\mathbf{e}_{34}) \\ \mathcal{I}_2: &\quad \tfrac{1}{4}(\mathbf{e}_{23}+\mathbf{e}_{14}),\ \tfrac{1}{4}(\mathbf{e}_{31}+\mathbf{e}_{24}),\ \tfrac{1}{4}(\mathbf{e}_{12}+\mathbf{e}_{34}), \end{aligned}$$

   each isomorphic to $\bigwedge^2 \mathbb{R}^3$. Show that there is no automorphism of the Lie algebra $\bigwedge^2 \mathbb{R}^4$ which permutes $\mathcal{V}, \mathcal{I}_1, \mathcal{I}_2$ cyclically or swaps $\mathcal{V}$ for $\mathcal{I}_1$ or $\mathcal{I}_2$.
7. In two dimensions we can place 4 circles of radius r inside a square of side $4r$, and put a circle of radius $(\sqrt{2}-1)r$ in the middle of the 4 circles. In three dimensions we can place 8 spheres of radius r inside a cube of side $4r$, and put a sphere of radius $(\sqrt{3}-1)r$ in the middle of the 8 circles. In n dimensions we can place 2^n spheres of radius r inside a hypercube of side $4r$, and put a sphere of radius $(\sqrt{n}-1)r$ in the middle of the 2^n spheres.

Dimensions 2 and 3 differ topologically: in dimension 3 one can see the middle sphere from outside the cube. Let the dimension be progressively increased. In some dimension the middle sphere actually emerges out of the hypercube. In some dimension the middle sphere becomes even bigger than the hypercube, in the sense that its volume is larger than the volume of the hypercube. Determine those dimensions.

Solutions

1. $\mathbf{e}_{1234}$, $\mathbf{e}_{12} + \mathbf{e}_{34}$.
3. $\cos\alpha\cos\beta + \mathbf{e}_{12}\sin\alpha\cos\beta + \mathbf{e}_{34}\cos\alpha\sin\beta + \mathbf{e}_{1234}\sin\alpha\sin\beta$.
5. The rotation angles are $\alpha = (|\mathbf{a}| + |\mathbf{b}|)/2$ and $\beta = (|\mathbf{a}| - |\mathbf{b}|)/2$, and

$$\frac{1}{2}(1+\mathbf{e}_{1234})\Big(\cos|\mathbf{a}| + \frac{\mathbf{A}}{|\mathbf{a}|}\sin|\mathbf{a}|\Big) + \frac{1}{2}(1-\mathbf{e}_{1234})\Big(\cos|\mathbf{b}| + \frac{\mathbf{B}}{|\mathbf{b}|}\sin|\mathbf{b}|\Big)$$
$$= \cos\alpha\cos\beta - \mathbf{e}_{1234}\sin\alpha\sin\beta$$
$$+\mathbf{C}\frac{\alpha - \beta\mathbf{e}_{1234}}{\alpha^2 - \beta^2}(\sin\alpha\cos\beta + \mathbf{e}_{1234}\cos\alpha\sin\beta).$$

7. In dimension 9 the middle sphere touches the surface of the hypercube, and in dimension 10 it emerges out of the hypercube. In dimension 1206 the volume of the middle sphere is larger than the volume of the hypercube.

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7
The Cross Product

The cross product is useful in many physical applications. It measures the angular velocity $\vec{\omega} = \vec{r} \times \vec{v}$ about O of a body moving at velocity $\vec{v}$ at the position P, $\vec{r} = \overrightarrow{OP}$. It is used to describe the torque $\vec{r} \times \vec{F}$ about O of a force $\vec{F}$ acting at $\vec{r}$. It also gives the force $\vec{F} = q\vec{v} \times \vec{B}$ acting on a charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$.

The usefulness of the cross product in three dimensions suggests the following questions: Is there a higher-dimensional analog of the cross product of two vectors in $\mathbb{R}^3$? If an analog exists, is it unique?

The first question is usually responded to by giving an answer to a modified question by explaining that there is a higher-dimensional analog of the cross product of $n-1$ vectors in $\mathbb{R}^n$. However, such a reply not only does not answer the original question, but also gives an incomplete answer to the modified question. In this chapter we will give a complete answer to the above questions and their modifications.

7.1 Scalar product in $\mathbb{R}^3$

The linear space $\mathbb{R}^3$ can be given extra structure by introducing the *scalar product* or *dot product*

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

for vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ in $\mathbb{R}^3$. The scalar product is scalar valued, $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$, and satisfies

$$\left.\begin{array}{l} (\mathbf{a}+\mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\ (\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) \end{array}\right\} \quad \text{linear in the first factor}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad \text{symmetric}$$

$$\mathbf{a} \cdot \mathbf{a} > 0 \quad \text{for} \quad \mathbf{a} \neq 0 \qquad \text{positive definite.}$$

Linearity with respect to the first argument together with symmetry implies that the scalar product is linear with respect to both arguments, that is, it is *bilinear*. The symmetric bilinear scalar valued product gives rise to the quadratic form

$$\mathbb{R}^3 \to \mathbb{R}, \quad \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \to \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2,$$

which makes the linear space $\mathbb{R}^3$ into a *quadratic space* $\mathbb{R}^3$. The quadratic form is positive definite, that is, $\mathbf{a}\cdot\mathbf{a} = 0$ implies $\mathbf{a} = 0$, which allows us to introduce the *length* [1] $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ of a vector $\mathbf{a} \in \mathbb{R}^3$. The real linear space $\mathbb{R}^3$ with a positive definite quadratic form on itself is called a *Euclidean space* $\mathbb{R}^3$. The length and the scalar product satisfy

$\lvert\mathbf{a}+\mathbf{b}\rvert \le \lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert$	triangle inequality
$\lvert\mathbf{a}\cdot\mathbf{b}\rvert \le \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$	Cauchy-Schwarz inequality

where the latter inequality gives rise to the concept of angle. The angle φ between two directions $\mathbf{a}$ and $\mathbf{b}$ is obtained from

$$\cos\varphi = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Thus, we can write the scalar product in the form

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\varphi,$$

a formula which is usually taken as a definition of the scalar product, although this requires prior introduction of the concepts of length and angle.

7.2 Cross product in $\mathbb{R}^3$

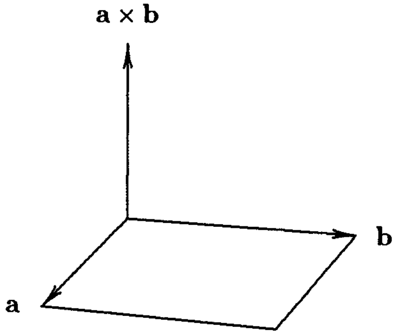
In the Euclidean space $\mathbb{R}^3$ it is convenient to introduce a vector valued product, the *cross product* $\mathbf{a}\times\mathbf{b} \in \mathbb{R}^3$ of $\mathbf{a},\mathbf{b} \in \mathbb{R}^3$, with the following properties:

$(\mathbf{a}\times\mathbf{b}) \perp \mathbf{a},\ (\mathbf{a}\times\mathbf{b}) \perp \mathbf{b}$	orthogonality
$\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\varphi$	length equals area
$\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b}$	right-hand system.

In other words, the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, its length is equal to the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as edges, and the vectors

1 The function $\mathbb{R}^3 \to \mathbb{R}$, $\mathbf{a} \to |\mathbf{a}|$ is a *norm* satisfying $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$, $|\mathbf{a}| = 0 \Rightarrow \mathbf{a} = 0$. Since this norm can be obtained from a scalar product, it satisfies the parallelogram law $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.

$\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are oriented according to the right hand rule.



The above definition results in the following multiplication rules:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2 \times \mathbf{e}_1,$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 = -\mathbf{e}_3 \times \mathbf{e}_2,$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_1 \times \mathbf{e}_3.$$

It is convenient to write the cross product in the form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The cross product is uniquely determined by

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0,\ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$	orthogonality
$\lvert\mathbf{a} \times \mathbf{b}\rvert^2 = \lvert\mathbf{a}\rvert^2 \lvert\mathbf{b}\rvert^2 - (\mathbf{a} \cdot \mathbf{b})^2$	Pythagorean theorem

together with the right hand rule. The Pythagorean theorem can be written using the *Gram determinant* as

$$|\mathbf{a} \times \mathbf{b}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}$$

which in coordinate form means *Lagrange's identity*

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2.$$

The cross product satisfies the following rules for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$:

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	antisymmetry
$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	interchange rule.

The antisymmetry of the cross product has a geometric meaning: the lack of symmetry measures how much the two directions diverge. The cross product is not associative, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, which results in an inconvenience in computation, because parentheses cannot be omitted. [2]

The cross product is dual to the exterior product of two vectors:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})\mathbf{e}_{123}.$$

Taking the exterior product of $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) - |\mathbf{a}|^2\mathbf{b}$ and $\mathbf{b}$ one finds that

$$\mathbf{a} \cdot \mathbf{b} = \frac{(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} \quad \text{for} \quad \mathbf{a} \not\parallel \mathbf{b},$$

that is, the scalar product can be recaptured from the cross product [you can also replace $\wedge$ by $\times$ in the above formula].

7.3 Cross product of $n-1$ vectors in $\mathbb{R}^n$

We can associate to three given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^4$ a fourth vector

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

which is orthogonal to the factors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and whose length is equal to the volume of the parallelepiped with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as edges, that is,

$$|\mathbf{a} \times \mathbf{b} \times \mathbf{c}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}.$$

The cross product $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^4$ is completely antisymmetric and obeys the interchange rule slightly modified:

$$(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c} \times \mathbf{d})$$

where $\mathbf{d} \in \mathbb{R}^4$. The oriented volume of the 4-dimensional parallelepiped with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as edges is the scalar

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}$$

multiplied by (the unit oriented volume) $\mathbf{e}_{1234}$.

2 The cross product is antisymmetric, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and satisfies the Jacobi identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$, which makes the linear space $\mathbb{R}^3$, with cross product on $\mathbb{R}^3$, a non-associative algebra, called a *Lie algebra*. The Jacobi identity can be verified using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

The cross product of three vectors in $\mathbb{R}^4$ is dual to the exterior product:

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = -(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})\mathbf{e}_{1234}$$

where the latter product is computed in the Clifford algebra $C\ell_4$.

In a similar manner we can introduce in n dimensions a cross product of $n-1$ factors. The result is a vector orthogonal to the factors, and the length of the vector is equal to the hypervolume of the parallelepiped formed by the factors.

7.4 Cross product of two vectors in $\mathbb{R}^7$

Is there a cross product in n dimensions with just two factors? If we require the cross product to be orthogonal to the factors and have length equal to the area of the parallelogram, then the answer is no, unless $n = 3$ or $n = 7$.

The cross product of two vectors in $\mathbb{R}^7$ can be defined in terms of an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ by antisymmetry, $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_j \times \mathbf{e}_i$, and

$$\begin{array}{lll}
\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_4, & \mathbf{e}_2 \times \mathbf{e}_4 = \mathbf{e}_1, & \mathbf{e}_4 \times \mathbf{e}_1 = \mathbf{e}_2, \\
\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_5, & \mathbf{e}_3 \times \mathbf{e}_5 = \mathbf{e}_2, & \mathbf{e}_5 \times \mathbf{e}_2 = \mathbf{e}_3, \\
\quad\vdots & \quad\vdots & \quad\vdots \\
\mathbf{e}_7 \times \mathbf{e}_1 = \mathbf{e}_3, & \mathbf{e}_1 \times \mathbf{e}_3 = \mathbf{e}_7, & \mathbf{e}_3 \times \mathbf{e}_7 = \mathbf{e}_1.
\end{array}$$

The above table can be condensed into the form

$$\mathbf{e}_i \times \mathbf{e}_{i+1} = \mathbf{e}_{i+3}$$

where the indices are permuted cyclically and translated modulo 7.

This cross product of vectors in $\mathbb{R}^7$ satisfies the usual properties, that is,

$$\begin{array}{ll}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0, \ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0 & \text{orthogonality} \\
|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 & \text{Pythagorean theorem}
\end{array}$$

where the second rule can also be written as $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \sphericalangle(\mathbf{a}, \mathbf{b})$. Unlike the 3-dimensional cross product, the 7-dimensional cross product does not satisfy the Jacobi identity, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} \neq 0$, and so it does not form a Lie algebra. However, the 7-dimensional cross product satisfies the Malcev identity, a generalization of Jacobi, see Ebbinghaus et al. 1991 p. 279.

In $\mathbb{R}^3$ the direction of $\mathbf{a} \times \mathbf{b}$ is unique, up to two alternatives for the orientation, but in $\mathbb{R}^7$ the direction of $\mathbf{a} \times \mathbf{b}$ depends on a 3-vector defining the cross product; to wit,

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \lrcorner \mathbf{v} \quad [\neq -(\mathbf{a} \wedge \mathbf{b})\mathbf{v}]$$

depends on

$$\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^7.$$

In the 3-dimensional space $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are in the same plane, but for the cross product $\mathbf{a} \times \mathbf{b}$ in $\mathbb{R}^7$ there are also other planes than the linear span of $\mathbf{a}$ and $\mathbf{b}$ giving the same direction as $\mathbf{a} \times \mathbf{b}$.

The 3-dimensional cross product is invariant under all rotations of $SO(3)$, while the 7-dimensional cross product is not invariant under all of $SO(7)$, but only under the exceptional Lie group G_2, a subgroup of $SO(7)$. When we let $\mathbf{a}$ and $\mathbf{b}$ run through all of $\mathbb{R}^7$, the image set of the simple bivectors $\mathbf{a} \wedge \mathbf{b}$ is a manifold of dimension $2\cdot 7-3 = 11 > 7$ in $\bigwedge^2 \mathbb{R}^7$, $\dim(\bigwedge^2 \mathbb{R}^7) = \frac{1}{2}7(7-1) = 21$, while the image set of $\mathbf{a} \times \mathbf{b}$ is just $\mathbb{R}^7$. So the mapping

$$\mathbf{a} \wedge \mathbf{b} \to \mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \,\lrcorner\, \mathbf{v}$$

is not a one-to-one correspondence, but only a method of associating a vector to a bivector.

The 3-dimensional cross product is the pure/vector part of the quaternion product of two pure quaternions, that is,

$$\mathbf{a} \times \mathbf{b} = \mathrm{Im}(\mathbf{ab}) \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathbb{H}.$$

In terms of the Clifford algebra $C\ell_3 \simeq \mathrm{Mat}(2, \mathbb{C})$ of the Euclidean space $\mathbb{R}^3$ the cross product could also be expressed as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{ab}\mathbf{e}_{123} \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset C\ell_3.$$

In terms of the Clifford algebra $C\ell_{0,3} \simeq \mathbb{H} \times \mathbb{H}$ of the negative definite quadratic space $\mathbb{R}^{0,3}$ the cross product can be expressed not only as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{ab}\mathbf{e}_{123} \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset C\ell_{0,3}$$

but also as [3]

$$\mathbf{a} \times \mathbf{b} = \langle \mathbf{ab}(1 - \mathbf{e}_{123}) \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset C\ell_{0,3}.$$

Similarly, the 7-dimensional cross product is the pure/vector part of the octonion product of two pure octonions, that is, $\mathbf{a} \times \mathbf{b} = \langle \mathbf{a} \circ \mathbf{b} \rangle_1$. The octonion algebra $\mathbb{O}$ is a norm-preserving algebra with unity 1, whence its pure/imaginary part is an algebra with cross product, that is, $\mathbf{a} \times \mathbf{b} = \frac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a})$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^7 \subset \mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$. The octonion product in turn is given by

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

3 This expression is also valid for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset C\ell_3$, but the element $1 - \mathbf{e}_{123}$ does not pick up an ideal of $C\ell_3$. Recall that $C\ell_3$ is simple, that is, it has no proper two-sided ideals.

for $a = \alpha + \mathbf{a}$ and $b = \beta + \mathbf{b}$ in $\mathbb{R} \oplus \mathbb{R}^7$. If we replace the Euclidean space $\mathbb{R}^7$ by the negative definite quadratic space $\mathbb{R}^{0,7}$, then not only

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

for $a, b \in \mathbb{R} \oplus \mathbb{R}^{0,7}$, but also

$$a \circ b = \langle ab(1 - \mathbf{v})\rangle_{0,1}$$

where $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^{0,7}$.

7.5 Cross products of k vectors in $\mathbb{R}^n$

If one reformulates the question about the existence of a cross product of two vectors in $\mathbb{R}^n$, and also allows $n-1$ factors, then one is led to a more general problem on the existence of a cross product of k factors in $\mathbb{R}^n$. If we were looking for a vector valued product of k factors in $\mathbb{R}^n$, then we should first formalize our problem by modifying the Pythagorean theorem, a candidate being the Gram determinant. A natural thing to do is to consider a vector valued product $\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k$ satisfying

$$\begin{array}{ll} (\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k) \cdot \mathbf{a}_i = 0 & \text{orthogonality} \\ |\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k|^2 = \det(\mathbf{a}_i \cdot \mathbf{a}_j) & \text{Gram determinant} \end{array}$$

where the second condition means that the length of $\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots \times \mathbf{a}_k$ equals the volume of the parallelepiped with $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ as edges.

The solution to this problem is that there are vector valued cross products in

3	dimensions with	2	factors
7	dimensions with	2	factors
n	dimensions with	$n-1$	factors
8	dimensions with	3	factors

and no others – except if one allows degenerate solutions, when there would also be in all even dimensions n, $n \in 2\mathbb{Z}$, a vector product with only one factor (and in one dimension an identically vanishing cross product with two factors).

The cross product of three vectors in $\mathbb{R}^8$ can be expressed as

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \,\lrcorner\, (\mathbf{w} - \mathbf{v}\mathbf{e}_8) = \langle(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})(1 - \mathbf{e}_{12\ldots8})\mathbf{w}\rangle_1$$

where

$$\begin{aligned} \mathbf{w} &= -(\mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713})\mathbf{e}_{12\ldots7} \\ &= \mathbf{e}_{1236} - \mathbf{e}_{1257} - \mathbf{e}_{1345} + \mathbf{e}_{1467} + \mathbf{e}_{2347} - \mathbf{e}_{2456} - \mathbf{e}_{3567} \end{aligned}$$

and $\mathbf{w} \in \bigwedge^4 \mathbb{R}^7 \subset \bigwedge^4 \mathbb{R}^8$.

The trivial cross product with one factor in an even number of dimensions rotates all vectors by 90°. Thus, let n be even and let $\mathbf{a}$ be the only factor of a trivial cross product with value $\mathbf{b}$, $|\mathbf{b}| = |\mathbf{a}|$, $\mathbf{b} \cdot \mathbf{a} = 0$. This can be accomplished by

$$\mathbf{b} = \mathbf{a} \lrcorner (\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 + \ldots + \mathbf{e}_{n-1}\mathbf{e}_n).$$

Exercises

1. Show that the cross product $\mathbf{a} \times \mathbf{r}$ can be represented by a matrix multiplication $A\mathbf{r} = \mathbf{a} \times \mathbf{r}$, where

$$A\mathbf{r} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

2. Express the rotation matrix e^A in terms of I, A and A^2. Hint: use the Cayley-Hamilton theorem, $A^3 + |\mathbf{a}|^2 A = 0$.
3. Express the rotated vector $e^A\mathbf{r}$ as a linear combination of $\mathbf{r}$, $\mathbf{a} \times \mathbf{r}$ and $(\mathbf{a} \cdot \mathbf{r})\mathbf{a}$. Hint: $A^2\mathbf{r} = (\mathbf{a} \cdot \mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r}$.
4. Compute the square of $\mathbf{w} = -\mathbf{v}\mathbf{e}_{12\ldots7} \in \bigwedge^4 \mathbb{R}^7$.
5. Show that $\frac{1}{8}(1 + \mathbf{w})$ is an idempotent of $\mathcal{C}\ell_7 \simeq \mathrm{Mat}(8, \mathbb{C})$.

Solutions

2. $e^A = I + \dfrac{A}{\alpha}\sin\alpha + \dfrac{A^2}{\alpha^2}(1 - \cos\alpha)$, where $\alpha = |\mathbf{a}|$.
3. $e^A\mathbf{r} = \cos\alpha\,\mathbf{r} + \dfrac{\sin\alpha}{\alpha}\mathbf{a} \times \mathbf{r} + \dfrac{1 - \cos\alpha}{\alpha^2}(\mathbf{a} \cdot \mathbf{r})\mathbf{a}$.
4. $\mathbf{w}^2 = 7 + 6\mathbf{w}$.

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