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characteristic function   (Definition)

Let X be a random variable. The *characteristic function* of X is a function $\varphi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi_X(t) = E e^{itX} = E\cos(tX) + iE\sin(tX),$$

that is, $\varphi_X(t)$ is the expectation of the random variable e^{itX}.

Given a random vector $\overline{X} = (X_1, \ldots, X_n)$, the characteristic function of $\overline{X}$, also called *joint characteristic function* of $X_1, \ldots, X_n$, is a function $\varphi_{\overline{X}} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\varphi_{\overline{X}}(t) = E e^{i \overline{t} \cdot \overline{X}},$$

where $\overline{t} = (t_1, \ldots, t_n)$ and $\overline{t} \cdot \overline{X} = t_1 X_1 + \cdots + t_n X_n$ (the dot product.)

Remark. If F_X is the distribution function associated to X, by the properties of expectation we have

$$\varphi_X(t) = \int_{\mathbb{R}} e^{itx} dF_X(x),$$

which is known as the Fourier-Stieltjes transform of F_X, and provides an alternate definition of the characteristic function. From this, it is clear that the characteristic function depends only on the distribution function of X, hence one can define the characteristic function associated to a distribution even when there is no random variable involved. This implies that two random variables with the same distribution must have the same characteristic function. It is also true that each characteristic function determines a unique distribution; hence the name, since it characterizes the distribution function (see property 6.)

Properties

1. The characteristic function is bounded by 1, i.e. $\left|\varphi_X(t)\right| \leq 1$ for all t;

2. $\varphi_X(0) = 1$;

3. $\overline{\varphi_X(t)} = \varphi_X(-t)$, where $\bar{z}$ denotes the complex conjugate of z;

4. φ_X is uniformly continuous in $\mathbb{R}$;

5. If X and Y are independent random variables, then $\varphi_{X+Y} = \varphi_X \varphi_Y$;

6. The characteristic function determines the distribution function; hence, $\varphi_X = \varphi_Y$ if and only if $F_X = F_Y$. This is a consequence of the *inversion* formula: Given a random variable X with characteristic function φ and distribution function F, if x and y are continuity points of F such that $x < y$, then

$$F(x) - F(y) = \frac{1}{2\pi} \lim_{s \to \infty} \int_{-s}^{s} \frac{e^{-itx} - e^{-ity}}{it} \varphi(t) dt;$$

7. A random variable X has a symmetrical distribution (i.e. one such that $F_X = F_{-X}$) if and only if $\varphi_X(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$;

8. For real numbers a, b, $\varphi_{aX+b}(t) = e^{itb} \varphi_X(at)$;

9. If $E|X|^n < \infty$, then φ_X has continuous n-th derivatives and

$$\frac{d^k \varphi_X}{dt^k}(t) = \varphi_X^{(k)}(t) = \int_{\mathbb{R}} (ix)^k e^{itx} dF_X(x), \quad 1 \leq k \leq n.$$

Particularly, $\varphi_X^{(k)}(0) = i^k E X^k$; characteristic functions are similar to moment generating functions in this sense.

Similar properties hold for joint characteristic functions. Other important result related to characteristic functions is the Paul Lévy continuity theorem.

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See Also: moment generating function

Other names: joint characteristic function

Cross-references: Paul Lévy continuity theorem, moment generating functions, derivatives, continuous, real numbers, inversion, independent, uniformly continuous, complex conjugate, bounded, clear, properties, distribution function, dot product, random vector, expectation, function, random variable
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