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Characteristic Function

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Given a subset A of a larger set, the characteristic function χ_A is defined to be identically one on A
Characteristic functions are sometimes denoted using the so-called Iverson bracket, and can be usef
is easier to say, for example, "the characteristic function of the primes" rather than repeating a given
function is a special case of a simple function.

The term characteristic function is used in a different way in probability, where it is denoted $\phi(t)$ an
transform of the probability density function *using Fourier transform parameters* $(a, b) = (1, 1)$,

$$
\begin{aligned}
\phi(t) &= \mathcal{F}_x[P(x)](t) = \int_{-\infty}^{\infty} e^{itx} P(x)\,dx \\
&= \int_{-\infty}^{\infty} P(x)\,dx + it \int_{-\infty}^{\infty} x\,P(x)\,dx + \frac{1}{2}(it)^2 \int_{-\infty}^{\infty} x^2 \\
&= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mu_k' \\
&= 1 + it\,\mu_1' - \frac{1}{2}t^2\,\mu_2' - \frac{1}{3!}it^3\,\mu_3' + \frac{1}{4!}t^4\,\mu_4' + \dots,
\end{aligned}
$$

where μ_n' (sometimes also denoted ν_n) is the nth moment about 0 and $\mu_0' \equiv 1$ (Abramowitz and St
1995). A statistical distribution is *not* uniquely specified by its moments, but is uniquely specified by

$$
P(x) = \mathcal{F}_t^{-1}[\phi(t)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\,dt
$$

(Papoulis 1984, p. 155).

The characteristic function can therefore be used to generate raw moments,

$$
\phi^{(n)}(0) \equiv \left[\frac{d^n \phi}{dt^n}\right]_{t=0} = i^n \mu_n'
$$

or the cumulants κ_n,

$$
\ln \phi(t) \equiv \sum_{n=0}^{\infty} \kappa_n \frac{(it)^n}{n!}.
$$

SEE ALSO: Cumulant, Iverson Bracket, Moment, Moment-Generating Function, Probability Density Fur
[Pages Linking Here]

Portions of this entry contributed by Todd Rowland

REFERENCES:

Abramowitz, M. and Stegun, I. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical* Dover, p. 928, 1972.

Kenney, J. F. and Keeping, E. S. "Moment-Generating and Characteristic Functions," "Some Examples of Moment-Genera Theorem for Characteristic Functions." §4.6-4.8 in *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand,

Morrison, K. E. "Cosine Products, Fourier Transforms, and Random Sums." *Amer. Math. Monthly* **102**, 716-724, 1995.

Papoulis, A. "Characteristic Functions." §5-5 in *Probability, Random Variables, and Stochastic Processes, 2nd ed.* New Yo 1984.

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