

Characteristic function

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Some mathematicians use the phrase **characteristic function** synonymously with *indicator function*. The indicator function of a subset A of a set B is the function with domain B , whose value is 1 at each point in A and 0 at each point that is in B but not in A .

In probability theory, the **characteristic function** of any probability distribution on the real line is given by the following formula, where X is any random variable with the distribution in question:

$$\phi_X(t) = \mathbf{E}(e^{itX}) = \int_{\Omega} e^{itx} dF_X(x) = \int_{-\infty}^{\infty} f_X(x) e^{itx} dx.$$

Here t is a real number, \mathbf{E} denotes the expected value and F is the cumulative distribution function. The last form is valid only when the probability density function f exists. The form preceding it is a Riemann-Stieltjes integral and is valid regardless of whether a density function exists.

If X is a vector-valued random variable, one takes the argument t to be a vector and tX to be a dot product.

A characteristic function exists for any random variable. More than that, there is a bijection between cumulative probability distribution functions and characteristic functions. In other words, two probability distributions never share the same characteristic function.

Given a characteristic function ϕ , it is possible to reconstruct the corresponding cumulative probability distribution function F :

$$F_X(y) - F_X(x) = \lim_{\tau \rightarrow +\infty} \frac{1}{2\pi} \int_{-\tau}^{+\tau} \frac{e^{-itx} - e^{-ity}}{it} \phi_X(t) dt.$$

In general this is an improper integral; the function being integrated may be only conditionally integrable rather than Lebesgue integrable, i.e. the integral of its absolute value may be infinite.

Characteristic functions are used in the most frequently seen proof of the central limit theorem.

Characteristic functions can also be used to find moments of random variable. Provided that n -th moment exists, characteristic function can be differentiated n times and

$$\mathbf{E}(X^n) = -i^n \phi_X^{(n)}(0) = -i^n \left[\frac{d^n}{dt^n} \phi_X(t) \right]_{t=0}.$$

Related concepts include the moment-generating function and the probability-generating function.

The characteristic function is closely related to the Fourier transform: the characteristic function of a distribution with density function f is proportional to the inverse Fourier transform of f . In fact, the probability distribution function is equal to the Fourier transform of the characteristic function (up to a constant of proportionality and assuming the integral is defined)

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(t) e^{-itx} dt.$$

Characteristic functions are particularly useful for dealing with functions of independent random variables. For example, if X_1, X_2, \dots, X_n is a sequence of independent (and not necessarily identically distributed) random variables, and

$$S_n = \sum_{i=1}^n a_i X_i,$$

where the a_i are constants, then the characteristic function for S_n is given by

$$\phi_{S_n}(t) = \phi_{X_1}(a_1 t) \phi_{X_2}(a_2 t) \cdots \phi_{X_n}(a_n t).$$

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