Characteristic Functions

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Characteristic Functions: The probabilistic name for the Fourier transform of the distribution of a random variable is the *Characteristic Function.* In other words, if X is a random variable with CDF F_X, then the characteristic function of X, or more properly of F_X, is defined for $t \in \mathbf{R}$ to be

$$\hat{F}_X(t) = E\left(e^{itX}\right).$$

Here, if $W = U + iV$ is a complex valued random variable, then $E(W) = E(U) + iE(V)$, assuming that U and V have finite expected values. Note that we still have the property that $|E(W)| \leq E(|W|)$.

Properties: Characteristic functions have similar properties to other transforms, such as generating functions, although some of them are more difficult to prove than for the case of generating functions. For example:

1. The characteristic function uniquely determines the distribution.
2. The characteristic function of the sum of independent random variables is the product of the characteristic functions of each of the random variables.
3. If $\{X_n\}$ is a sequence of i.i.d. random variables with common characteristic function $\hat{F}$ and N is an *independent* non-negative integer valued random variable with probability generating function g, then the characteristic function of the random sum $S_N = X_1 + X_2 + \cdots + X_N$ is given by $g(\hat{F}(t))$.
4. If X has finite expected value, then $E(X) = -i\hat{F}'_X(0)$. More generally, if $E(|X|^k) < \infty$, then $E(X^k) = (-i)^k \hat{F}_X^{(k)}(0)$.
5. Change of location and scale: Suppose $\sigma > 0$ and $\mu \in \mathbf{R}$. Then
$$\hat{F}_{\sigma X+\mu}(t) = E\left(e^{it(\sigma X+\mu)}\right) = e^{it\mu}\hat{F}_X(\sigma t).$$
6. Convergence in distribution is equivalent to pointwise convergence of the corresponding characteristic functions.

Examples: Here are a few characteristic functions that can be very useful.

1. Suppose X is Bernoulli with probability of success p. Then

$$\hat{F}_X(t) = e^{it}p + (1-p) = 1 - p(1-e^{it}).$$

2. Consequently, if Y is binomial(n,p) then

$$\hat{F}_Y(t) = \left[e^{it}p + (1-p)\right]^n = \left[1 - p(1-e^{it})\right]^n.$$

3. Suppose X is Poisson with parameter $\lambda > 0$. Then

$$\hat{F}_X(t) = e^{-\lambda}e^{\lambda e^{it}} = e^{-\lambda(1-e^{it})}.$$

4. Suppose that X is Gamma with parameters $\alpha > 0$ and $\lambda > 0$. Then

$$\hat{F}_X(t) = \lambda^\alpha/(\lambda - it)^\alpha.$$

5. Suppose that X has a standard Cauchy distribution. Then

$$\hat{F}_X(t) = e^{-|t|}.$$

6. Suppose Z is standard normal, *i.e.* $N(0,1)$, so that Z has density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

for $x \in \mathbf{R}$. Then

$$\hat{F}_Z(t) = e^{-t^2/2} = \sqrt{2\pi}\varphi(t).$$

7. If X is normal with mean μ and variance σ^2, i.e. $X \sim N(\mu, \sigma^2)$, then $X \sim \sigma Z + \mu$ where $Z \sim N(0,1)$. Consequently,

$$\hat{F}_X(t) = e^{it\mu}\hat{F}_Z(\sigma t) = e^{it\mu}e^{-\sigma^2 t^2/2}.$$