Induced and higher-dimensional stable independence (Part II)

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After last week's summary of recent work on category-theoretic stable independence, we turn to more practical matters:

- We show that, if $\mathcal{K}$ has a weakly stable, $\aleph$-continuous independence notion, and stable independence on a nicely embedded subcategory, it has stable independence.
- Using this result and recent work ([KMA],[MA1],[MA2]) we show that a large number of familiar categories from algebra have a stable independence notion.
- Returning to the crucial special case from last week—$\mathcal{K}$ locally presentable—we show that the existence of stable independence implies excellence; that is, stable independence in all dimensions.

We begin with a short review...

We think of independence on an abstract category $\mathcal{K}$ as follows:

Definition

An independence notion $\downarrow$ on $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\downarrow$ is **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

As I suggested last week, an independence notion—particularly if weakly stable—can be thought of as a replacement for pushouts that may be lacking in $\mathcal{K}$, or, more tellingly, $\mathcal{K}_{\mathcal{M}}$.

Locality is achieved by requiring accessibility of arrow category $\mathcal{K}_\downarrow$:

- Objects: Morphisms $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\mathop{\smile}\limits^{|}$-independent square

$$\begin{tikzcd} M' \arrow[r] & N' \\ M \arrow[u] \arrow[r] \arrow[ur, phantom, "\mathop{\smile}\limits^{|}"] & N \arrow[u] \end{tikzcd}$$

Definition

1. *We say that $\mathop{\smile}\limits^{|}$ is λ-**continuous** if $\mathcal{K}_\downarrow$ is closed under λ-directed colimits.*
2. *We say that $\mathop{\smile}\limits^{|}$ is λ-**accessible** if $\mathcal{K}_\downarrow$ is λ-accessible.*
3. *We say $\mathop{\smile}\limits^{|}$ is λ-**stable** if it is weakly stable and λ-accessible.*

Given a category $\mathcal{K}$ and family of morphisms $\mathcal{M}$, recall that the induced subcategory $\mathcal{K}_{\mathcal{M}}$ has a natural candidate for $\perp$:

Definition

We say a square

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

*in $\mathcal{K}$ is $\mathcal{M}$-**effective** if*

1. *all morphisms are in $\mathcal{M}$,*
2. *the pushout of $M_1 \leftarrow M_0 \rightarrow M_2$ exists, and*
3. *the induced map from the pushout to M_3 is in $\mathcal{M}$.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

To force these squares to form a nice independence relation, we need a few additional properties:

Definition

Let $\mathcal{K}$ be a category.

1. *We say that $\mathcal{M}$ is* **coherent** *if whenever $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, $f \in \mathcal{M}$.*
2. *We say that $\mathcal{M}$ is a* **coclan** *if pushouts of morphisms in $\mathcal{M}$ exist, and $\mathcal{M}$ is closed under pushouts.*
3. *We say $\mathcal{M}$ is* **almost nice** *if it is a coherent coclan, and* **nice** *if, in addition, it is closed under retracts.*

Proposition

If $\mathcal{M}$ is almost nice, the $\mathcal{M}$-effective squares give a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.

Let us assume that $\mathcal{K}$ is an accessible category with all morphisms monomorphisms (hence, morally speaking, a μ-AEC).

As we saw last week, failure of the order property implies the existence of stable independence on a cofinal subcategory. We will see presently that this also follows from Galois-stability.

It is natural, then, to ask: Given a subcategory $\mathcal{L}$ of $\mathcal{K}$ that has stable independence, under what conditions on

- the category $\mathcal{K}$, and
- the embedding $\mathcal{L} \hookrightarrow \mathcal{K}$

can we infer the existence of stable independence on $\mathcal{K}$?

Conditions on $\mathcal{K}$

We will not be able to manufacture stable independence on $\mathcal{K}$ out of whole cloth.

In particular, $\mathcal{K}$ must have an independence notion that is

1. weakly stable, and
2. $\aleph_0$-continuous.

The latter property implies only that $\mathcal{K}_\downarrow$ is closed under directed colimits—lacking the local character that would come with, say, $\aleph_0$-accessibility.

So we assume a lot, but much less than stability.

Conditions on $\mathcal{L} \hookrightarrow \mathcal{K}$

The term *cofinal* has already been bandied about. To be precise:

Definition

We say that a functor $F : \mathcal{L} \to \mathcal{K}$ is *cofinal* if for any $K \in \mathcal{K}$ and any finite sequence $(FL_i \stackrel{f_i}{\to} K)_{i \in I}$, there is $L \in \mathcal{L}$, $K \stackrel{g}{\to} FL$, and $(FL_i \stackrel{Fg_i}{\to} FL)_{i \in I}$ such that $Fg_i = f_i \circ g$ for all $i \in I$. A subcategory is cofinal if the inclusion is cofinal.

This is weaker than the usual category-theoretic notion.

- If $\mathcal{L}$ is a full subcategory of $\mathcal{K}$ and for every $K \in \mathcal{K}$ there is a morphism $f : M \to L$ with $L \in \mathcal{L}$, then $\mathcal{L}$ is a cofinal subcategory of $\mathcal{K}$ in the above sense.
- The category of λ-saturated models of a μ-AEC is a cofinal subcategory.

Conditions on $\mathcal{L} \hookrightarrow \mathcal{K}$

In addition to cofinality, we will need to ensure that the embedding $\mathcal{L} \hookrightarrow \mathcal{K}$ plays well with the accessible structure on $\mathcal{K}$:

Definition

We say that a subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is *accessibly embedded* if

- $\mathcal{L}$ is a full subcategory, and
- $\mathcal{L}$ is closed under λ-directed colimits in $\mathcal{K}$ for some λ.

This is, somewhat confusingly, different from requiring that the embedding $\mathcal{L} \to \mathcal{K}$ is *accessible*—that involves preservation of colimits.

Theorem

Let $\mathcal{K}$ be an accessible category with all morphisms monomorphisms, and let $\mathcal{L}$ be an accessibly-embedded, cofinal subcategory of $\mathcal{K}$. If:

- *$\mathcal{K}$ has an $\aleph_0$-continuous weakly stable independence notion.*
- *$\mathcal{L}$ has a stable independence notion.*

Then $\mathcal{K}$ has a stable independence notion.

Proof: (Sketch) Let $\downarrow$ be the $\aleph_0$-continuous, weakly stable notion on $\mathcal{K}$. We must show that $\mathcal{K}_{\downarrow}$ is accessible.

By $\aleph_0$-continuity, $\mathcal{K}$ has directed colimits, meaning $\mathcal{L}$ has directed bounds. Since the restriction of $\downarrow$ to $\mathcal{L}$ will be weakly stable (check!), the canonicity theorem ensures that it is, in fact, stable.

This implies that the restriction of $\downarrow$ to $\mathcal{L}$ is accessible, meaning $\mathcal{L}$ is accessible. Take μ with $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{L}_\downarrow$ all μ-accessible.

Let $\mathcal{K}_\mu$ be the subcategory on the μ-presentable objects of $\mathcal{K}$. Let $\mathcal{K}^*$ be the set of all μ-directed colimits of $\mathcal{K}_\mu$-morphisms in $\mathcal{K}_\downarrow$. It suffices to show that $\mathcal{K}^* = \mathcal{K}^2$, the full arrow category. Notice that $\mathcal{L}^2 \subseteq \mathcal{K}^*$, in any case.

To achieve this, we show:

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

We show sufficiency, but give only brief sketches of the proofs of (i)-(iii).

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

By cofinality, there is $p : M^* \to L$ with $L \in \mathcal{L}$, and $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$. By (i), $phg \in \mathcal{K}^*$. Since $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, and we are done.

Proof of (i): Given composable f and g in $\mathcal{K}^*$, we can decompose them as μ-directed colimits and, with a little fiddling, ensure that (enough of) the arrows in these colimits are composable—this gives $gf \in \mathcal{K}^*$.

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

By cofinality, there is $p : M^* \to L$ with $L \in \mathcal{L}$, and $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$. By (i), $phg \in \mathcal{K}^*$. Since $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, and we are done.

Proof of (ii): Quite finicky. Skip...

(i) $\mathcal{K}^*$ is closed under composition.

(ii) $\mathcal{K}^*$ is left cancellable.

(iii) If $M \in \mathcal{K}$, there is a morphism $M \to N$ in $\mathcal{K}^*$ with $N \in \mathcal{L}$.

Sufficient: for any $f : M \to N$ in $\mathcal{K}^2$, (iii) implies there is $g : M \to M'$ in $\mathcal{K}^*$ with $M' \in \mathcal{L}$. There is $h : M' \to M^*$ with M λ-saturated, $\lambda > \mu$. Then there is $t : N \to M^*$ with $tf = hg$.

By cofinality, there is $p : M^* \to L$ with $L \in \mathcal{L}$, and $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$. By (i), $phg \in \mathcal{K}^*$. Since $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, and we are done.

Proof of (iii): By contradiction. Uses well μ-filtrability of $\mathcal{K}$ ([LRV]); that is, objects are colimits of smooth chains.

Recall from last week that stable independence implies Galois-stability:

Theorem

Let $\mathcal{K}$ be a (μ-)AEC with a stable independence relation. For any α, there is a proper class of cardinals S_α such that for any $\lambda \in S_\alpha$ and $M \in \mathcal{K}_\lambda$, $|ga\text{-}S^{<\alpha}(M)| = \lambda$.

There is no hope, of course, that Galois-stability will correspond to the existence of stable independence: the latter implies amalgamation, for one thing.

Given how well understood Galois-stability is, though, and how many nice categories have this property, we should pursue whatever limited converses we can.

We will, of course, use the theorem just discussed. But of critical importance, too, is:

Lemma

Let $\mathcal{K}$ be an AEC. If

1. *$\mathcal{K}$ has the amalgamation property,*
2. *$\mathcal{K}$ is Galois-stable, and*
3. *types in $\mathcal{K}$ are $< \aleph_0$-short over models,*

then there is a stable independence notion on a full, cofinal subcategory of $\mathcal{K}$—consisting of sufficiently saturated models.

Proof: (Sketch) Essentially the same as the construction of stable independence over saturated models of a stable first-order theory: $\aleph_0$-shortness stands in for compactness (cf. [V]).

Thus Galois-stability, in an AEC with amalgamation and $\aleph_0$-shortness, buys us much of what we need to derive existence of stable independence on the AEC itself.

We still require a weakly stable, $\aleph_0$-continuous independence notion. But we already have an excellent candidate:

Lemma

Let $\mathcal{K}$ be a category, and $\mathcal{M}$ an almost nice family of morphisms.

- *$\mathcal{K}_\mathcal{M}$ has a weakly stable independence notion consisting of the $\mathcal{M}$-effective squares.*
- *If $\mathcal{M}$ is closed under directed colimits, this independence notion is $\aleph_0$-continuous [LRV2].*

So, if our AEC is formed by taking an almost nice, direct-colimit closed set of morphisms, we are in business. In all of the algebraic examples to follow, happily, this is the case.

We now stand on the shoulders of giants—Kucera and Mazari-Armida—who have done all of the model-theoretic heavy lifting. First, a template:

Proposition

For any ring with unit R, the category of left R-modules, R-$\mathbf{Mod}_{pure}$, has a stable independence notion.

Proof: By [KMA], R-$\mathbf{Mod}_{pure}$ forms an AEC, has amalgamation, is stable, and types are $< \aleph_0$-short over models. So, by the lemma, there is stable independence on a cofinal, full subcategory.

By inspection, or the fact that pure monomorphisms are the left half of a coherent WFS, pure monomorphisms are almost nice, closed under directed colimits. So $\mathcal{M}$-effective squares are weakly stable, $\aleph_0$-continuous.

The main theorem then yields stable independence on R-$\mathbf{Mod}_{pure}$.

We note that this is a special case of the result of [LPRV]: the latter was obtained by other means, but discovered by these.

The same template can be applied to, e.g. the following categories of modules. Here R is an integral domain.

1. Torsion R-modules with pure monomorphisms.
2. R-divisible modules (for any nonzero m and nonzero $r \in R$, there is n with $m = rn$) with pure monomorphisms.

Similarly, we obtain stable independence on many categories of groups, including:

1. Abelian groups and (pure) monos.
2. (Reduced) Torsion-free abelian groups with pure monos.
3. Abelian p-groups with (pure) monos.

We turn now to something which is still in search of applications: higher-dimensional stable independence on an abstract category.

Definition

Let $\mathcal{K}$ be a category. For $n \geq 1$, we define an *n-dimensional stable independence relation* on $\mathcal{K}$, Γ, and its induced category $\mathcal{K}_\Gamma$ by induction on n:

- We say Γ is a 1-dimensional stable independence notion on $\mathcal{K}$ just in case it is $\mathrm{Mor}(\mathcal{K})$. In this case, define $\mathcal{K}^\Gamma = \mathcal{K}$.
- An $(n+1)$-dimensional stable independence relation on $\mathcal{K}$ consists of a pair (Γ_n, Γ), where
    1. Γ_n is an n-dimensional stable independence relation on $\mathcal{K}$.
    2. Γ is a stable independence notion on $\mathcal{K}^{\Gamma_n}$
- Given $(n+1)$-dimensional $\Gamma_{n+1} = (\Gamma_n, \Gamma)$ on $\mathcal{K}$, define $(\mathcal{K}^{\Gamma_{n+1}} = \mathcal{K}^{\Gamma_n})^\Gamma$, whose objects are morphisms of $\mathcal{K}^{\Gamma_n}$ and whose morphisms are Γ-independent squares (that is, $\mathcal{K}_\downarrow$ with $\perp = \Gamma$).

That is too much to internalize in one sitting, of course.

(As an exercise, one might check that 2-dimensional stable independence notions correspond to stable independence notions as already defined.)

The best-case—and presumably rare—scenario is the following:

Definition

We say that a category $\mathcal{K}$ is *excellent* if for all $n \geq 1$, $\mathcal{K}$ has an n-dimensional stable independence relation Γ_n such that $\mathcal{K}^{\Gamma_n}$ has directed colimits.

We return to our favorite special case: $\mathcal{K}$ locally presentable.

Theorem

Let $\mathcal{K}$ be a locally presentable category, and let $\mathcal{M}$ be a nice, accessible, $\aleph_0$-continuous class of morphisms in $\mathcal{K}$. If $\mathcal{K}_{\mathcal{M}}$ has a stable independence relation, it is excellent.

Theorem

Let $\mathcal{K}$ be a locally presentable category, and let $\mathcal{M}$ be a nice, accessible, $\aleph_0$-continuous class of morphisms in $\mathcal{K}$. If $\mathcal{K}_\mathcal{M}$ has a stable independence relation, it is excellent.

Proof: (Sketch) We wish to proceed by induction on dimension.

Recall that, under these hypotheses, $\mathcal{K}_\mathcal{M}$ has stable independence just in case $\mathcal{M}$ is cofibrantly generated.

So really, the inductive step involves showing that, given the above assumptions, the class of $\mathcal{M}$-effective morphisms in $\mathcal{K}^2$—call it $\mathcal{M}!$—is well-behaved in exactly the same ways:

- $\mathcal{M}!$ is cofibrantly generated in $\mathcal{K}^2$.
- $\mathcal{M}!$ is nice, $\aleph_0$-continuous, and accessible.

Nearly everything is just bookkeeping, aside from showing $\mathcal{M}!$ is a coclan and that it is cofibrantly generated (easier via stability!).

There are plenty of open questions, some of which have already occurred naturally in this discussion:

- What happens if we weaken different conditions in the definition of stable independence, particularly uniqueness?
- Do superstability, simplicity, etc. admit clean category-theoretic characterizations, e.g. via properties of WFSs?
- How much use can we get out of stable independence and related constructions, e.g. independent sequences, in an abstract category?
- In what other contexts can we get stability via the induced route taken here?
- What is excellence good for in an abstract category? Does it correspond to any existing notions?

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