# Induced and higher-dimensional stable independence (Part I)

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Over the course of two talks, we aim to provide an introduction to stable independence in the context of an abstract category, building toward applications in algebra.

- Today: We consider basic definitions and motivation, canonicity, and connections with model theory. In case the underlying category is locally presentable, we also indicate a deep connection with homotopy theory.
- Next week: We discuss ways in which stable independence can be pushed upward from a subcategory, ensuring existence in a host of categories of groups and modules. Time permitting, we discuss excellence in the category-theoretic context.

All of the results considered here are joint with Jiří Rosický and Sebastien Vasey, spanning several papers: [LRV1], [LRV2], and [LRV3].

One of the central features of model theory—classical or abstract—is that it only entertains injective maps, or *monomorphisms*.

## Example

*Consider $T_{ab}$, the theory of abelian groups. The category of models of $T_{ab}$, $\mathbf{Mod}(T_{ab})$, has the same objects as* **Ab** *but is decidedly non-full: however we construe it, its morphisms will certainly be monos.*

Regardless of the level of generality (or specificity) at which we work, the model-theoretic toolkit works only if we make this fundamental restriction.

That toolkit is exceedingly powerful, but problems arise when we restrict to monomorphisms: category theory is much harder.

Recall that a *pushout* of a span $M_1 \leftarrow M_0 \to M_2$ is a commutative square

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

with the property that for any other completion of the span, $M_1 \to Q \leftarrow M_2$, there is a unique map $P \to Q$ so that everything commutes. The pushout is a minimal amalgam, roughly speaking.

### Fact

*The category* $\mathbf{Ab}$ *has pushouts;* $\mathbf{Mod}(T_{ab})$ *does not, as the induced maps will not be monos!*

**Basic problem:** given a category $\mathcal{K}$ and family of $\mathcal{K}$-morphisms $\mathcal{M}$, how much is lost in passing to $\mathcal{K}_{\mathcal{M}}$, the subcategory of $\mathcal{K}$ whose morphisms are precisely those in $\mathcal{M}$?

For the moment, we consider **very** nice $\mathcal{K}$.

## Definition

*For $\lambda$ a regular cardinal, we say that a category $\mathcal{K}$ is* **locally $\lambda$-presentable** *if*

1. *$\mathcal{K}$ has all colimits.*
2. *There is a set of $\lambda$-presentable objects, and every object of $\mathcal{K}$ is a $\lambda$-directed colimit thereof.*

This covers, e.g. **Set** , **Ab**, $R$-**Mod**, and **Str**$(\Sigma)$, where presentability corresponds, roughly, to cardinality/presentation size.

In general, passing to $\mathcal{K}_{\mathcal{M}}$ expels us from the paradise of locally presentable categories, leaving us with, if we are lucky, accessibility.

### Definition

*For $\lambda$ a regular cardinal, we say a category $\mathcal{K}$ is $\lambda$-**accessible** if*

1. *$\mathcal{K}$ has all $\lambda$-directed colimits.*
2. *There is a set of $\lambda$-presentable objects, and every object of $\mathcal{K}$ is a $\lambda$-directed colimit thereof.*

That is, we may lose some colimits, including pushouts.

### Fact

*Say a category $\mathcal{C}$ is accessible with all morphisms mono (...). If $\mathcal{C}$ has pushouts, it is small.*

So if we engineer $\mathcal{K}_{\mathcal{M}}$ to be nice, we lose pushouts. Such is life.

We take the view, perhaps controversially, that stable nonforking is best understood as a stand-in for the vanished pushouts.

This is extremely ahistorical...

Version 1: Fix a theory $T$, monster model $\mathfrak{C}$. We say the type of a tuple $\bar{a} \in \mathfrak{C}$ over a model $B$ does not fork over $C \subseteq B$ if the type over $C$ has the same complexity, i.e. Morley rank. Notation:

$$\bar{a} \overset{(\mathfrak{C})}{\underset{C}{\perp\!\!\!\perp}} B$$

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Version 2: Again, given a theory $T$ and monster model $\mathfrak{C}$, we say

$$A \underset{C}{\overset{(\mathfrak{C})}{\downarrow}} B$$

if the type of any $\bar{a} \in A$ over $B$ does not fork over $C$. One can think of this as a kind of independence relation: $A$ is independent from $B$ over $C$.

One can think of $\downarrow$ as an abstract ternary relation, and axiomatize stable (or *simple*) independence directly.

We take the view, perhaps controversially, that stable nonforking is best understood as a stand-in for the vanished pushouts.

This is extremely ahistorical...

Version 3: In AECs, we can only work over models, and may not have a monster model. We end up with $\downfree$ as a quaternary relation

$$M_1 \overset{M_3}{\underset{M_0}{\downfree}} M_2$$

axiomatized as before, [BGKV]. In particular, we are picking out a family of diagrams of strong embeddings of the form

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] \arrow[ur, phantom, "\downfree"] & M_2 \arrow[u] \end{tikzcd}$$

Idea: Do this in an arbitrary category $\mathcal{K}$.

## Definition

*An independence notion $\perp$ on $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\perp$ is* **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

## Fact

*If $\perp$ is weakly stable, these squares satisfy the usual cancellation property of pushouts.*

We must impose a locality condition—accessibility now appears.

Consider the category $\mathcal{K}_{\downarrow}$:

- Objects: Morphisms $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\downarrow$-independent square

$$\begin{tikzcd} M' \arrow[r] & N' \\ M \arrow[u] \arrow[r] \arrow[ur, phantom, "\downarrow"] & N \arrow[u] \end{tikzcd}$$

## Definition

1. *We say that $\downarrow$ is $\lambda$-**continuous** if $\mathcal{K}_{\downarrow}$ is closed under $\lambda$-directed colimits.*
2. *We say that $\downarrow$ is $\lambda$-**accessible** if $\mathcal{K}_{\downarrow}$ is $\lambda$-accessible.*
3. *We say $\downarrow$ is $\lambda$-**stable** if it is weakly stable and $\lambda$-accessible.*

Returning to the basic framework, i.e. $\mathcal{K}$ a category, $\mathcal{M}$ a class of morphisms, there is a natural candidate for stable independence:

## Definition

*We say a square*

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

*in $\mathcal{K}$ is $\mathcal{M}$-**effective** if*

1. *all morphisms are in $\mathcal{M}$,*
2. *the pushout of $M_1 \leftarrow M_0 \rightarrow M_2$ exists, and*
3. *the induced map from the pushout to $M_3$ is in $\mathcal{M}$.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

To force these squares to form a nice independence relation, we need a few additional properties:

## Definition

*Let $\mathcal{K}$ be a category.*

1. *We say that $\mathcal{M}$ is* **coherent** *if whenever $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, $f \in \mathcal{M}$.*
2. *We say that $\mathcal{M}$ is a* **coclan** *if pushouts of morphisms in $\mathcal{M}$ exist, and $\mathcal{M}$ is closed under pushouts.*
3. *We say $\mathcal{M}$ is* **almost nice** *if it is a coherent coclan, and* **nice** *if, in addition, it is closed under retracts.*

## Proposition

*If $\mathcal{M}$ is almost nice, the $\mathcal{M}$-effective squares give a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*

For the next few slides, unless otherwise indicated, we consider the ways in which stable independence notions manifest themselves in $\mu$-*abstract elementary classes* (or $\mu$-*AECs*).

### Note

*A $\mu$-AEC is just like an AEC, but with a $\mu$-ary signature and closed under $\mu$-directed unions, rather than unions of chains.*

### Fact (BGLRV)

*$\mu$-AECs and accessible categories with all morphisms monomorphisms are, morally speaking, exactly the same thing.*

This makes them an ideal test case for evaluating the model-theoretic content of our stable independence notions.

Some care is required when translating back to an element- and type-oriented description: we must take a kind of closure ($\overline{\perp}$).

The essential bridge is provided by the *witness property*:

## Definition

*Let $\theta$ be an infinite cardinal. We say $\perp$ has the* **right $(<\theta)$-witness property** *if $M_1 \underset{M_0}{\overset{M_3}{\perp}} M_2$ holds whenever*

*$M_0 \prec_{\mathcal{K}} M_i \prec_{\mathcal{K}} M_3$, $i = 1, 2$, and $M_1 \underset{M_0}{\overset{M_3}{\overline{\perp}}} A$ for all $A \subseteq UM_2$ with $|A| < \theta$. Similarly,* **left $(<\theta)$-witness property**, **$(<\theta)$-witness property**.

## Theorem

*A reasonable independence notion on a $\mu$-AEC is accessible iff it has the witness property and local character (in the usual sense).*

### Fact (BG, essentially)

*The witness property holds in any fully tame and short $\mu$-AEC.*

With this rephrasing, model-theoretic consequences become much clearer. For example:

### Theorem

*Let $\mathcal{K}$ be a $\mu$-AEC with a stable independence relation. Then:*

1. *(Galois-stability) For any $\alpha$, there is a proper class of cardinals $S_\alpha$ such that for any $\lambda \in S_\alpha$ and $M \in \mathcal{K}_\lambda$, $|ga\text{-}S^{<\alpha}(M)| = \lambda$.*
2. *(Tameness) For any $\alpha$, there is a cardinal $\lambda$ such that $< \alpha$-types are $\lambda$-tame.*

What of the order property? Symmetry? Canonicity?

We consider the version of the order property introduced by Shelah in the context of AECs:

### Definition

*A $\mu$-AEC $\mathcal{K}$ has the $\alpha$-**order property of length** $\theta$ if there exists $M \in \mathcal{K}$ and a sequence $\langle \bar{a}_i \mid i < \theta \rangle$ with*

1. $\bar{a}_i \in {}^{\alpha}UM$ *for all* $i < \theta$, *and*
2. *for all* $i_0 < j_0 < \theta$, *ga-tp*$(\bar{a}_{i_0}\bar{a}_{j_0}/\emptyset, M) \neq$ *ga-tp*$(\bar{a}_{j_0}\bar{a}_{i_0}/\emptyset, M)$

*We say $\mathcal{K}$ has the **order property** if there exists $\alpha$ such that for all $\theta$, $\mathcal{K}$ has the $\alpha$-order property of length $\theta$.*

One would expect the existence of a stable independence relation to imply failure of the order property...

## Theorem

*If $\mathcal{K}$ is a $\mu$-AEC with a stable independence notion, $\mathcal{K}$ does not have the order property.*

In fact, the same is true even if we drop the assumption that the independence notion has the witness property, provided that $\mathcal{K}$ has *chain bounds*:

## Definition

We say a category $\mathcal{K}$ has **chain bounds** if every chain has an upper bound (but not necessarily a union/colimit).

[An important lesson here, and in the canonicity theorem below, is that this weakening of unions of chains is almost always sufficient.]

There is a partial converse, but it is *very* partial:

### Theorem

*Assume Vopěnka's Principle (VP). Let $\mathcal{K}$ be a $\mu$-AEC with chain bounds, and let $\kappa$ be strongly compact. If $\mathcal{K}$ does not have the order property, then the $\kappa$-AEC of locally $\kappa$-model-homogeneous models of $\mathcal{K}$ has a stable independence relation.*

### Note

*If we assume $\mathcal{K}$ has amalgamation, VP is not necessary.*

### Corollary

*Assume VP. Let $\mathcal{K}$ be a $\mu$-AEC with chain bounds. Then $\mathcal{K}$ does not have the order property iff there is a stable independence notion on a cofinal sub-$\lambda$-AEC.*

An argument very similar to that in [BGKV] gives canonicity of stable independence in $\mu$-AECs.

More remarkably, it holds in far greater generality:

### Theorem (LRV2)

*Let $\mathcal{K}$ be a category with chain bounds, and let $\overset{1}{\downarrow}$ and $\overset{2}{\downarrow}$ be independence notions with existence and uniqueness such that:*

1. $\overset{1}{\downarrow}$ *is (right) monotonic.*
2. $\overset{2}{\downarrow}$ *is transitive and (right) accessible.*

*Then $\overset{1}{\downarrow} = \overset{2}{\downarrow}$. So, in particular, $\mathcal{K}$ has at most one stable independence notion.*

Granted: the existence of an accessible independence notion $\overset{2}{\downarrow}$ implies $\mathcal{K}$ is accessible. But we are not back in the realm of $\mu$-AECs: here the morphisms **need not be monomorphisms**!

This argument, in [LRV2], resembles that of [BGKV] but is, of necessity, element-free.

Though confined to an appendix, it deserves a lecture all its own: it shows, among other things, that independent sequences can be developed and put to use in an arbitrary (accessible) category.

One expects this to have interesting consequences in algebra...

We return to the special case considered at the start: $\mathcal{K}$ is a locally presentable category, and $\mathcal{M}$ a family of morphisms.

### Note

*Recall that if $\mathcal{M}$ is almost nice, $\mathcal{K}_{\mathcal{M}}$ has a weakly stable independence notion (given by $\mathcal{M}$-effective squares).*

When is this notion $\aleph_0$-continuous ($\mathcal{K}_{\mathcal{M},\downarrow}$ closed under directed colimits)? When is it $\aleph_0$-accessible ($\mathcal{K}_{\mathcal{M},\downarrow}$ finitely accessible) and therefore very stable?

While existence and uniqueness seem to be the thornier issues in model theory, it is these properties that are most problematic here.

In this special case, though, there is an easy (and altogether surprising!) answer.

We veer sharply in the direction of algebraic topology. Recall:

### Note

*In* **Top**, *CW-complexes are built inductively by gluing on new cells along their boundaries, $S^{n-1} \to D^n$. The corresponding morphisms are constructed in similar fashion...*

Gluing corresponds to pushing out along some $S^{n-1} \to D^n$.

The inductive construction corresponds to transfinite composition.

So we are concerned with the maps *cellularly generated* by the set $\{S^{n-1} \to D^n \,:\, n \in \omega\}$.

Being generated in this way from a **set** of morphisms is an important smallness condition...

## Definition

*Let $X$ be a family of morphisms in a category $\mathcal{K}$. Recall:*

1. $\mathrm{Po}(X)$ *is the closure of $X$ under pushouts.*
2. $\mathrm{Tc}(X)$ *is the closure under transfinite composition.*
3. $\mathrm{Rt}(X)$ *is the closure under retracts.*
4. $\mathrm{cell}(X) = \mathrm{Tc}(\mathrm{Po}(X))$
5. $\mathrm{cof}(X) = \mathrm{Rt}(\mathrm{cell}(X))$

Under certain circumstances, we can dispense with retracts.

## Definition

*We say that a set of morphisms $\mathcal{M}$ in $\mathcal{K}$ is* **cofibrantly generated** *if $\mathcal{M} = \mathrm{cof}(X)$, $X$ a* **set** *of morphisms.*

## Theorem

*Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ nice and $\aleph_0$-continuous. The following are equivalent:*

1. *$\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.*
2. *$\mathcal{M}$-effective squares form a stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*
3. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

## Proof.

(1) $\Rightarrow$ (2): By canonicity. $\square$

## Theorem

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3. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

## Proof.

(2) $\Rightarrow$ (3): Take $\lambda$ such that $\mathcal{K}_{\mathcal{M},\downarrow}$ and $\mathcal{K}$ are $\lambda$-accessible, consider

$$\mathcal{M}_\lambda = \mathcal{M} \cap \mathbf{Pres}_\lambda(\mathcal{K})^{\to}.$$

One can show that $\mathcal{M} = \mathrm{cof}(\mathcal{M}_\lambda)$. □

## Theorem

*Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ nice and $\aleph_0$-continuous. The following are equivalent:*

1. *$\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.*
2. *$\mathcal{M}$-effective squares form a stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*
3. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

## Proof.

(3) $\Rightarrow$ (1): Say $\mathcal{M} = \mathrm{cof}(X)$, and $\lambda$ such that everyone involved is $\lambda$-accessible, domains and codomains of morphisms in $X$ are $\lambda$-presentable. Show class $\mathcal{M}^*$ of $\lambda$-directed colimits of maps in $\mathcal{M}_\lambda$ (in $\mathcal{K}_{\mathcal{M},\downarrow}$) is exactly $\mathcal{M}$. Need elimination of retracts, [MRV]. □

## Definition

*A* **weak factorization system** *(or WFS) in a category $\mathcal{K}$ consists of a pair of classes of morphisms $(\mathcal{M}, \mathcal{N})$ such that:*

1. *Any morphism $h$ of $\mathcal{K}$ can be written as $h = gf$, where $f \in \mathcal{M}$ and $g \in \mathcal{N}$.*
2. *Morphisms in $\mathcal{M}$ and $\mathcal{L}$ satisfy certain (nonunique) lifting properties: $\mathcal{M} = {}^{\square}\mathcal{N}$ and $\mathcal{N} = \mathcal{M}^{\square}$.*

## Examples

*In* **Set***, (monos, epis), as would expect. Also: (epis, monos).*

By Quillen's small object argument, if $\mathcal{K}$ is locally presentable, and $\mathcal{M}$ cofibrantly generated, then $(\mathcal{M}, \mathcal{M}^{\square})$ is a WFS on $\mathcal{K}$!

# Model Categories

To carry out homotopy theory in a category $\mathcal{K}$, we need a *model structure*, consisting of:

1. $\mathrm{Cof} \subseteq \mathrm{Mor}(\mathcal{K})$, the cofibrations—nice inclusions, roughly.
2. $\mathrm{Fib} \subseteq \mathrm{Mor}(\mathcal{K})$, the fibrations—nice surjections, roughly.
3. $\mathcal{W} \subseteq \mathrm{Mor}(\mathcal{K})$, the weak equivalences—standing in for homotopy equivalences.

subject to the condition (among others) that:

- $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are WFSs.

Combinatorial model structures, in particular, are those where the class of acyclic cofibrations, $\mathrm{Cof} \cap \mathcal{W}$, is cofibrantly generated.

## Fact

*If $(\mathcal{M}, \mathcal{N})$ is a coherent WFS—that is, $\mathcal{M}$ is coherent—then $\mathcal{M}$ is nice and $\aleph_0$-continuous.*

## Corollary

*If $(\mathcal{M}, \mathcal{N})$ is a coherent WFS on locally presentable $\mathcal{K}$, TFAE:*

1. *$\mathcal{K}_{\mathcal{M}}$ has stable independence.*
2. *$\mathcal{M}$ is cofibrantly generated (and accessible).*

So, modulo a few important technicalities, subcategories $\mathcal{K}_{\mathcal{M}}$ with stable independence are in bijective correspondence with cofibrantly generated WFSs!

**[BG]** W. Boney and R. Grossberg, *Forking in short and tame AECs*, Ann. Pure Appl. Log. **168**(8), pp. 1517-1551 (2017).

**[BGKV]** W. Boney, R. Grossberg, A. Kolesnikov, and S. Vasey, *Canonical forking in AECs*, Ann. Pure Appl. Log. **167**(7), pp. 590-613 (2016).

**[BGLRV]** W. Boney, R. Grossberg, M. Lieberman, J. Rosický, and S. Vasey, *$\mu$-Abstract lementary classes and other generalizations*, J. Pure Appl. Alg. **220**(9), 3048-3066 (2016).

**[LRV1]** *Forking independence from the categorical point of view*, Adv. Math. **346**, pp. 719-772 (2020).

**[LRV2]** Cellular categories and stable independence. Submitted, arXiv:1904.05691.

**[LRV3]** Induced and higher-dimensional stable independence. Submitted, arXiv:2011.13962.

**[MRV]** M. Makkai, J. Rosický, and L. Vokřínek, *On a fat small object argument*, Adv. Math. **254**, pp. 49-68 (2014).