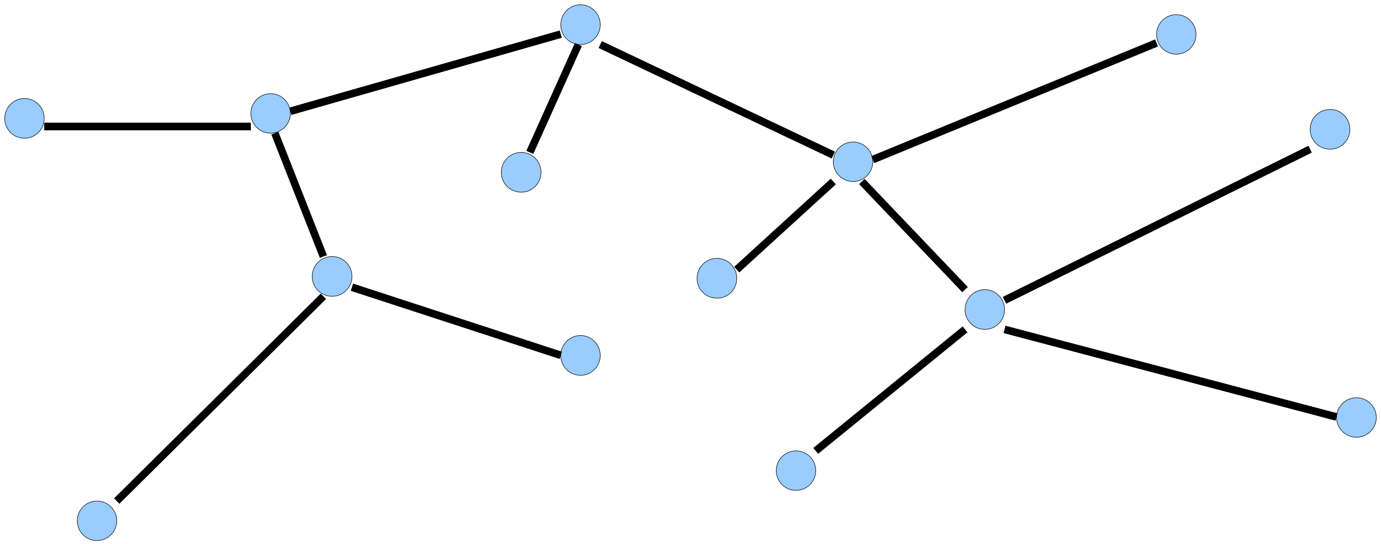
A connected graph with no circles is called a tree



We will prove a number of statements on graphs. As it is, each of them might itself be used to define a tree:

1. In a tree, there is a unique simple path between every pair of nodes.

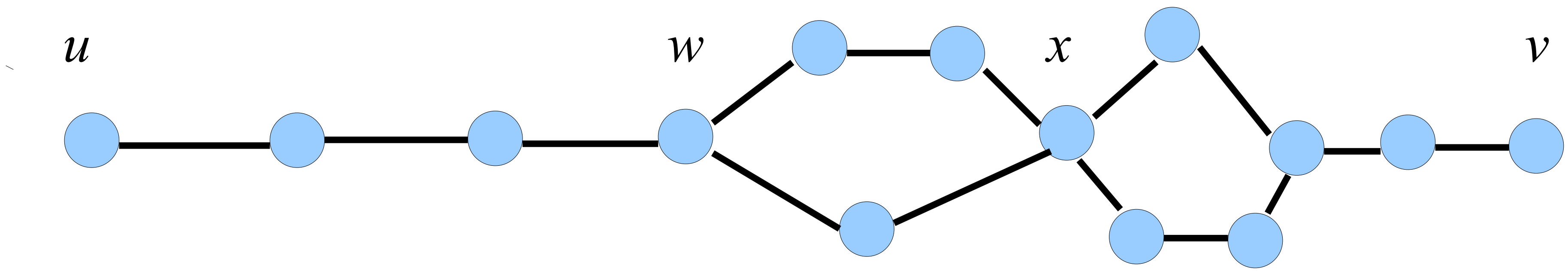
Let u and v be two nodes in a tree $T=(N,E)$.

Let $P_1(u,v)$ be a simple path between the nodes u and v and $P_2(v,w)$ a simple path between the nodes v and w, and let the sets of their edges be distinct. Then by $P_1 + P_2$ we will denote the simple path between u and w created by connecting P and Q in a natural way.

Let P' and P'' be two different paths between u and v. Then we can write:

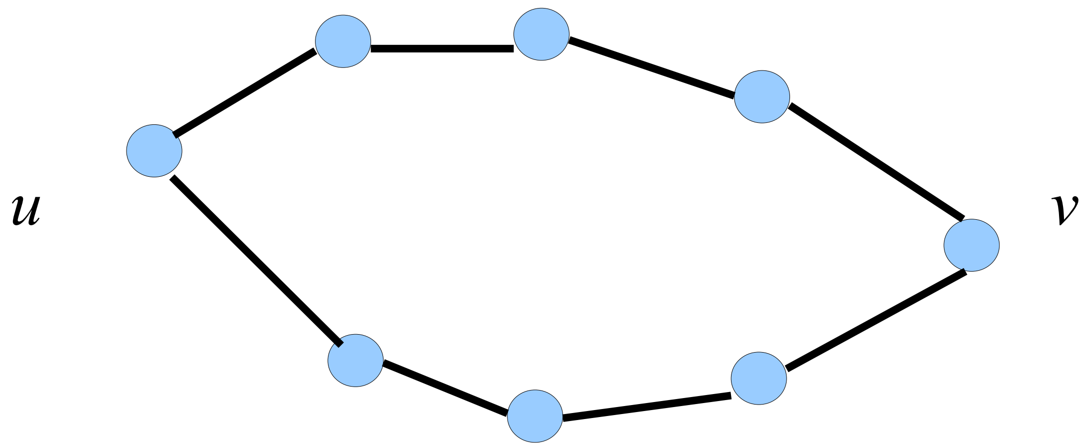
$$P'=P(u,w)+Q'(w,x)+R'(x,v) \quad P''=P(u,w)+Q''(w,x)+R''(x,v)$$

where the edges of the paths Q' and Q'' are disjunct sets. Here, the paths P, R' and R'' may also be empty, but Q' and Q'' are both non-empty with their edges forming disjunct sets. Clearly, $Q'(w,x)+Q'''(x,w)$ is a circle where $Q'''(x,w)$ has the same set of edges as $Q''(w,x)$. However, this is not possible since T is a tree.



2. If there is a unique simple path between every pair of nodes in a graph G, then G is a tree.

Suppose that G is not a tree, then there is a circle with at least two nodes u and v and, clearly, between such two nodes, there are at least two distinct paths.



3. Every edge in a tree T is a bridge.

Let e be an edge in a tree T between nodes u and v. If, after deleting e, there were still a path in the remaining graph between u and v, there would be two distinct paths in T between u and v, which is a contradiction since, by the above assertion, there would be a circle in T.

4. If G is a connected graph such that every edge e in G is a bridge, then G is a tree.

Suppose that G is not a tree. Then there is a circle C in G. Let e be an edge contained in C. By deleting e from G, we obtain graph G'. Since e is a bridge, G' is no longer connected. Let p, q be any two nodes in G. There is a path P in G between p and q. If P does not contain e, then P is also a path between p and q in the (disconnected) graph G'. If P does contain e, we take the path in G between p and q replacing e by the part of the path along circle C that does not contain e. Clearly, we again obtain a path and thus also a simple path between p and q. This means that G' is connected, which is a contradiction.

5. A tree T with n nodes has n – 1 edges.

This is clearly true for a tree T with one node. Let every tree with m nodes, $1 \leq m < n$, has $m - 1$ edges. Let e be an edge between nodes u and w in a tree T with n nodes. Then e is a bridge and, deleting it, we obtain a subgraph T' of T with two components H and H'. These components are clearly trees with k and k' nodes respectively where $k + k' = n$. This means that $k < n, k' < n$. By the induction hypothesis, H has $k - 1$ edges and H' has $k' - 1$ edges. Thus T' has $n - 2$ edges and T has $n - 1$ edges.

6. Any connected graph G with n nodes and $n-1$ edges is a tree.

Suppose that a graph $G=(N,E)$ with n nodes and $n-1$ edges is not a tree. Then there is an edge e which is not a bridge. By deleting e from G, we obtain again a connected graph. Clearly, in this way, after a finite number of steps, we arrive at a subgraph $H=(N,F)$ in which every edge is a bridge. Thus, by 4 and 5 above, H has exactly $n-1$ edges. Considering that $|F|<|E|$, this is a contradiction.

7. Any graph $G=(N,E)$ with n nodes and $n-1$ edges that contains no circles is connected and thus a tree.

Suppose that G is not connected. Let its components be $G_{1,} G_{2,} \ldots, G_r$. Being connected and containing no circle, each G_i is a tree and, as such, has n_i nodes and $n_i - 1$ edges by 5 above. Thus the total number of edges in G is $n_1 + n_2 + \cdots + n_r - r = n - r$. But G has exactly $n-1$ edges, which means that G has only one component and is connected.

8a Let T be a tree. If a new edge is added to T creating a graph T ', then T ' contains exactly one circle.

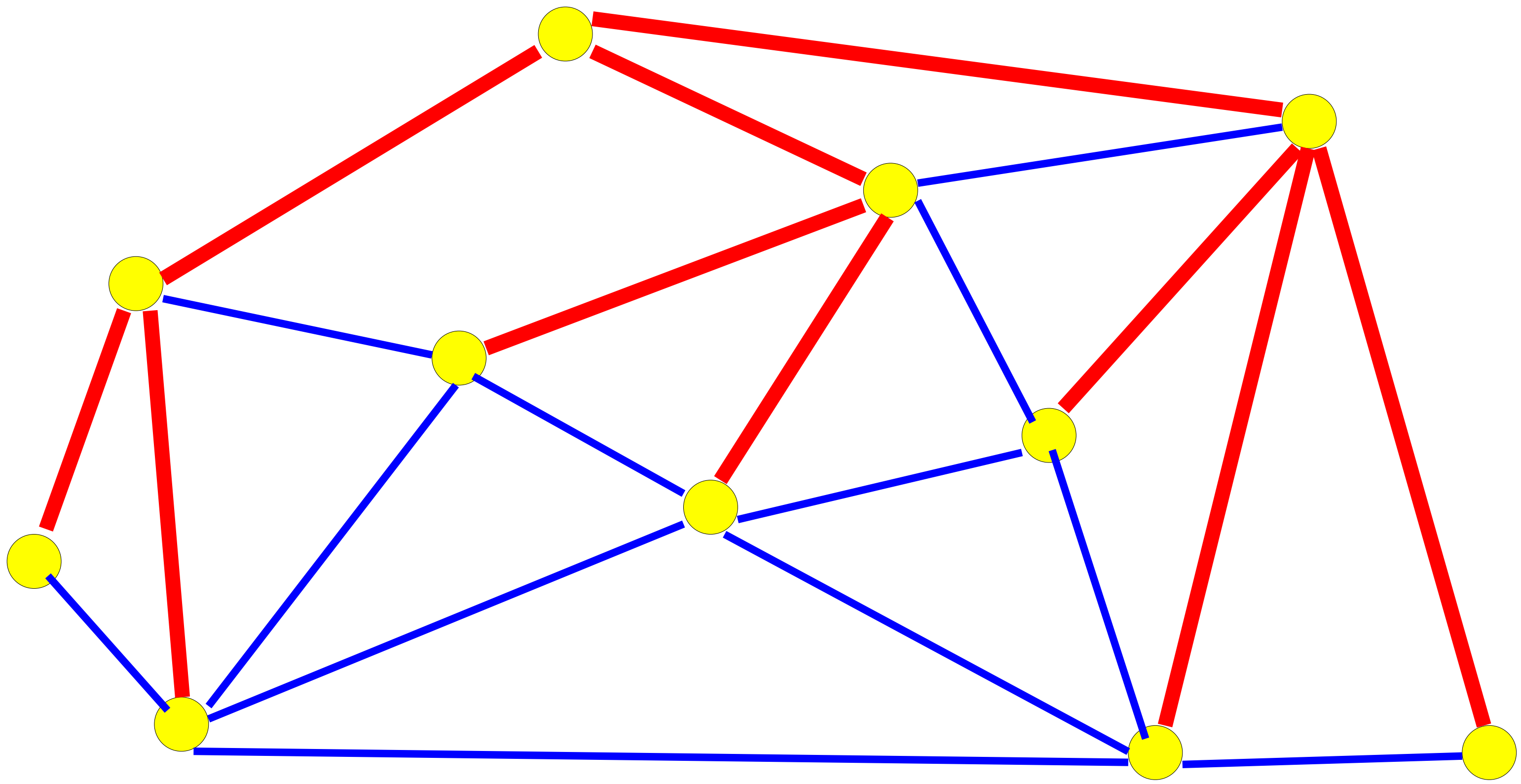
8b If, adding an edge to a graph G that contains no circle always results in a graph G ' that contains exactly one circle, then G is a tree.

To prove *8b* suppose that G is not a tree. Then it is not connected. This means that there is a pair of nodes p, q such that there is no path between them and thus the addition of the new edge $\{p, q\}$ cannot create a circle, which is a contradiction.

A subgraph T of a graph G with n nodes is called a ***spanning tree*** in G if

(a) T is a tree and

(b) T has n nodes.



Clearly, a graph G is connected if and only if it has a spanning tree.

THEOREM 1

Let G be a simple graph with n nodes. If a subgraph H of G has n nodes and has any two of the below properties, then it also has the third property.

(a) H is connected,

(b) H has $n-1$ *eges,*

(c) H contains no circle.

THEOREM 2

In a tree T with at least two nodes there are at least two nodes with a degree equal to 1.

Denote by P a simple path in T with a maximum number of edges. Since the number of edges in is finite, such a path always exists. Denote the two distinct end-nodes of P by p and q respectively. It is easy to see that node p can only be connected by an edge with the succeeding node in P while q can only be connected by an edge with the preceding node in P. If there were another edge connecting, say, p with another node, this would mean that either P is not maximal or T contains a circle. In both cases, this leads to a contradiction.

THEOREM 3 (Cayley)

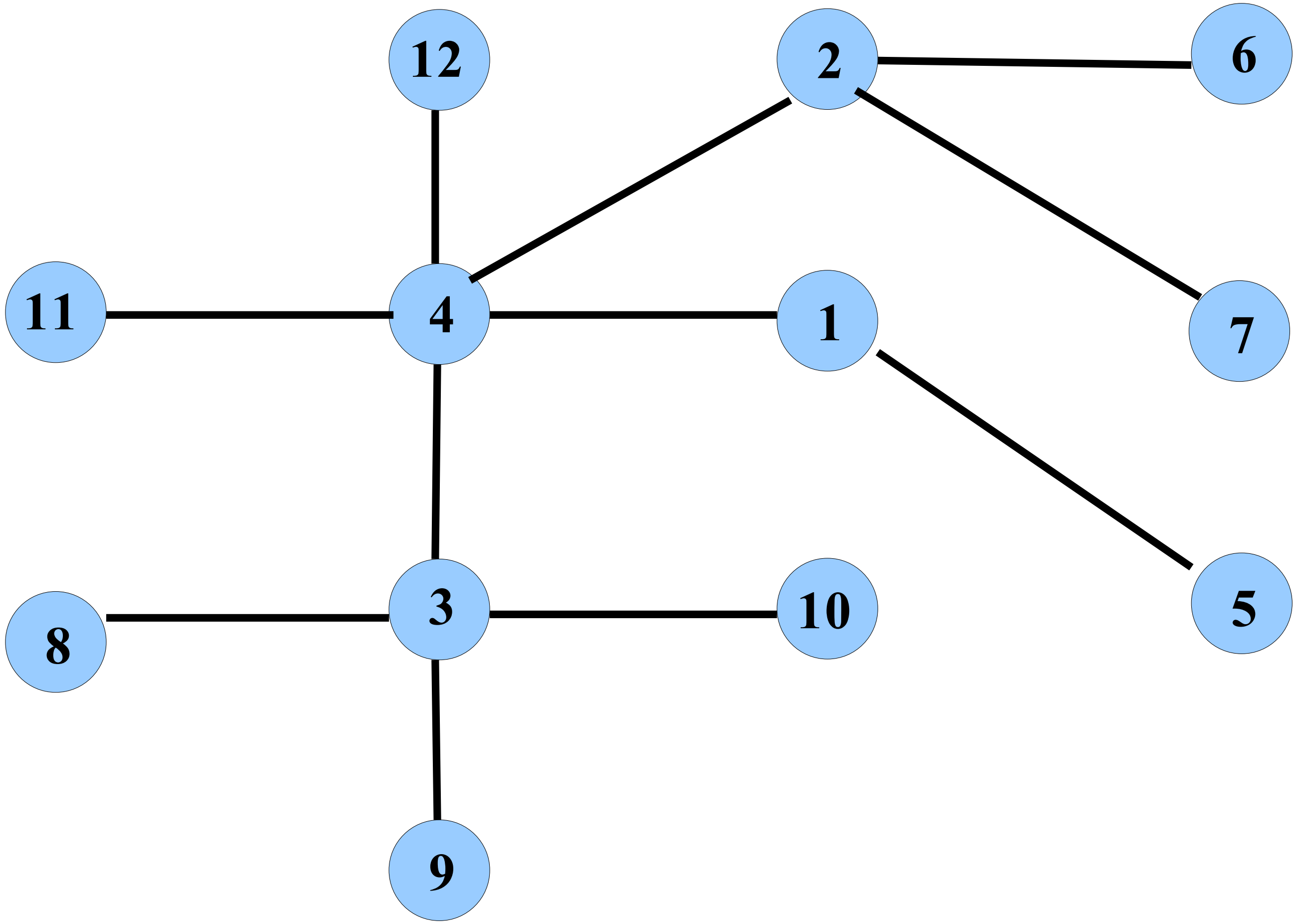
Let K_n with $n \geq 2$ be a complete graph. Then K_n has n^{n-2} spanning trees.

Denote by N the set $\{1,2,\ldots,n\}$ and let the nodes of K_n be labelled by the elements of N, that is, $1,2,\ldots,n$. Define a set T as follows:

$T = \{(k_1, k_2, \ldots, k_{n-2}) \mid k_i \in N, 1 \leq i \leq n\}$. It is not difficult to prove that T has exactly n^{n-2} elements. We will now establish a one-to-one correspondence between T and the set of all spanning trees of K_n. In the first part of the proof, for every spanning tree of K_n, we will find a unique $(n-2)$-tuple $(k_1, k_2, \ldots, k_{n-2})$ and then, using an arbitrary $(n-2)$-tuple $(k_1, k_2, \ldots, k_{n-2})$, we will construct a unique spanning tree of K_n.

Let S be a spanning tree of K_n. Denote by W_1 the sequence of distinct nodes of S with degrees equal to 1 arranged in an ascending order by their labels. Let w_1 be the first element in such a sequence. Let s_1 be the unique node adjacent in S to the node labelled w_1. Denote by S_1 the tree obtained by deleting w_1 from S and find a sequence W_2 of distinct nodes of S_1 with degrees equal to 1 arranged in an ascending order by their labels. Let w_2 be the first element is such a sequence. Let s_2 be the unique node adjacent in S_1 to the node labelled w_2. Continuing this process, we finally get an $(n-2)$-tuple $(s_{1,} s_{2,} \ldots, s_{n-2})$. Clearly, such an $(n-2)$-tuple is unique thanks to arranging the sequences $W_{1,} W_{2,} \ldots, W_{n-2}$ in ascending order. The following example will illustrate this procedure:

The below spanning tree



will produce the 10-tuple $(1,4,2,2,4,3,3,3,4,4)$

On the contrary, consider an $(n-2)$-tuple $T=(s_{1,}s_{2,}\ldots,s_{n-2})$ of labels in N. Arrange the nodes of K_n in an ascending order by their labels and define:

$$v_1=\min\{v_i|v_i\notin\{T\}\},$$

$$v_2=\min\{v_i|v_i\notin(\{T\}-\{s_1\}\cup\{v_1\})\}$$

$$\vdots$$

$$v_{n-2}=\min\{v_i|v_i\notin(\{T\}-\{s_{1,}s_{2,}\ldots,s_{n-3}\}\cup\{v_{1,}v_{2,}\ldots,v_{n-3}\})\}$$

where min defines the node with the smallest label. Now construct a tree by taking the nodes of K_n and joining by an edge each node labelled by s_i with the node labelled by v_i, $1\le i\le n-2$ and, finally, joining by an edge the two remaining nodes that are not among the nodes $v_{1,}v_{2,}\ldots,v_{n-2}$.