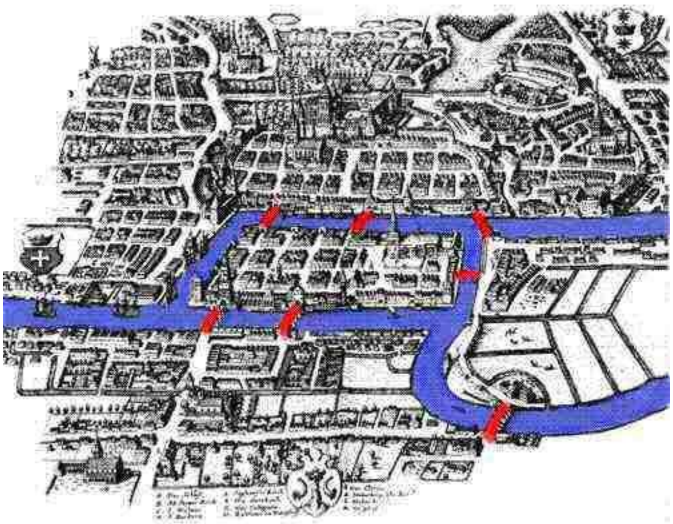
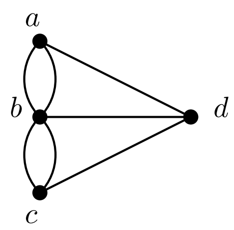
Section 9

Eulerian and Hamiltonian Graphs

Graph theory began with Euler's study of a particular problem: the Seven Bridges of Königsberg. During the eighteenth century the city of Königsberg (in East Prussia) was divided into four sections (including the island of Kneiphof) by the Pregel river. Seven bridges connected these regions and it was said that residents spent their Sunday walks trying to find a starting point so that they could walk about the city, cross each bridge exactly once, and return to their starting point.



To apply graph theory to this, let us represent each of the four sections of the city by a node in a graph and represent each bridge by an edge:



(This is an undirected multigraph.)

Note that the degrees of the vertices are the following values:

$$\rho(a) = \rho(c) = \rho(d) = 3, \qquad \rho(b) = 5.$$

The question we seek to ask is the following:

> Is there a circuit (a closed walk) that traverses every edge in the graph once?

Euler established that the answer to this question depended upon the number of vertices of odd degree in the graph.

We make the following definition:

Definition 9.1 A graph $\Gamma = (V, E)$ is called *Eulerian* if there is a circuit in Γ that passes through every vertex $v \in V$ and that traverses every edge of Γ exactly once.

A weakening is the following:

Definition 9.2 A graph $\Gamma = (V, E)$ is called *semi-Eulerian* if there is a walk in Γ that passes through every vertex $v \in V$ and that traverses every edge of Γ exactly once.

(In a semi-Eulerian graph, we do not require that we end up back where we started!)

Theorem 9.3 *Let $\Gamma = (V, E)$ be a connected graph. Then Γ is Eulerian if and only if every vertex has even degree.*

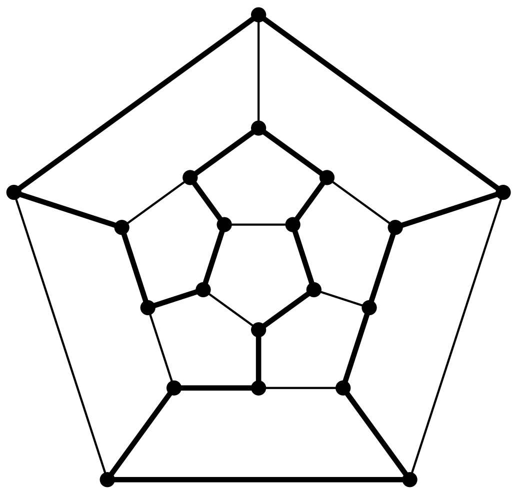
Corollary 9.4 *A connected graph $\Gamma = (V, E)$ is semi-Eulerian if and only if Γ has at most two vertices of odd degree.*

The graph for the Seven Bridges of Königsberg has four vertices of odd degree. Consequently it is neither Eulerian nor semi-Eulerian: the people of Königsberg were wasting their time on Sunday afternoons trying to find such a route!

Hamiltonian paths and cycles

In 1859, the Irish mathematician Sir William Rowan Hamilton developed a game that he sold to a Dublin toy manufacturer. The game consisted of a wood regular dodecahedron with the twenty corner points (vertex) labelled with the names of prominent cities. The object of the game was to find a circuit along the edges of the solid so that each city on the circuit exactly once.

We represent the solid by a graph: the vertices of the graph correspond to the vertices of the solid and the edges similarly correspond:



Definition 9.5 Let $\Gamma = (V, E)$ be a graph. A *Hamiltonian circuit* is a circuit which passes through every vertex exactly once (with only the first and last vertex being a repeat).

A graph is called *Hamiltonian* if it possesses a Hamiltonian circuit.

Unsolved Problem: What is a necessary and sufficient condition for a graph to be Hamiltonian?

This question appears to be extremely difficult to solve. The following gives a sufficient condition:

Theorem 9.6 (Dirac 1952) *Let $\Gamma = (V, E)$ be a simple graph with n vertices and suppose $\rho(v) \geqslant n/2$ for every vertex v. Then Γ is Hamiltonian.*

(We can see easily that this is not a necessary condition. The dodecahedron graph corresponding to Hamilton's original game has $n = 20$, $\rho(v) = 3$ for every vertex v, yet the graph *is* Hamiltonian.)

PROOF: Suppose Γ is not Hamiltonian. If we were to add more edges to Γ, then eventually we would have to create a graph which is Hamiltonian. Therefore we may add a number of edges to Γ and create a simple graph Γ' which is not Hamiltonian, but for which the addition of a single further edge gives a Hamiltonian graph. Note that $\rho'(v) \geqslant \rho(v) \geqslant n/2$ where $\rho'(v)$ denotes the degree of the vertex v in the new graph Γ'.

Let $\{v_1, v_2\}$ be the edge which when added creates a Hamiltonian circuit. This circuit necessarily has the form

$$v_1 \to v_2 \to v_3 \to \cdots \to v_{i-1} \to v_i \to \cdots \to v_n \to v_1.$$

Now if for some i (with $3 \leqslant i \leqslant n$) there is an edge $\{v_2, v_i\}$ and an edge $\{v_1, v_{i-1}\}$ in Γ', then we can create a new Hamiltonian circuit

$$v_2 \to \underbrace{v_i \to v_{i+1} \to \cdots \to v_n} \to v_1 \to \underbrace{v_{i-1} \to v_{i-2} \to \cdots \to v_3 \to v_2}.$$

This circuit does not involve the edge $\{v_1, v_2\}$ and so exists in the graph Γ'. This contradicts the assumption that Γ' is not Hamiltonian.

Hence for each of $i = 3, \ldots, n$, it is not the case that there is both an edge $\{v_1, v_{i-1}\}$ and an edge $\{v_2, v_i\}$. Let $A' = (a'_{kl})$ be the adjacency matrix of Γ'. This assertion is that

$$a'_{1,i-1} + a'_{2i} \leqslant 1 \qquad \text{for } 3 \leqslant i \leqslant n.$$

Let us sum over all i:

$$\sum_{j=2}^{n-1} a'_{1j} + \sum_{i=3}^{n} a'_{2j} \leqslant n - 2$$

so

$$\sum_{j=1}^{n} a'_{1j} + \sum_{i=1}^{n} a'_{2j} \leqslant n - 1.$$

(Note $a'_{11}, a'_{12}, a'_{21}, a'_{22} = 0$, as Γ' is simple and does not have $\{v_1, v_2\}$ as an edge.) Hence

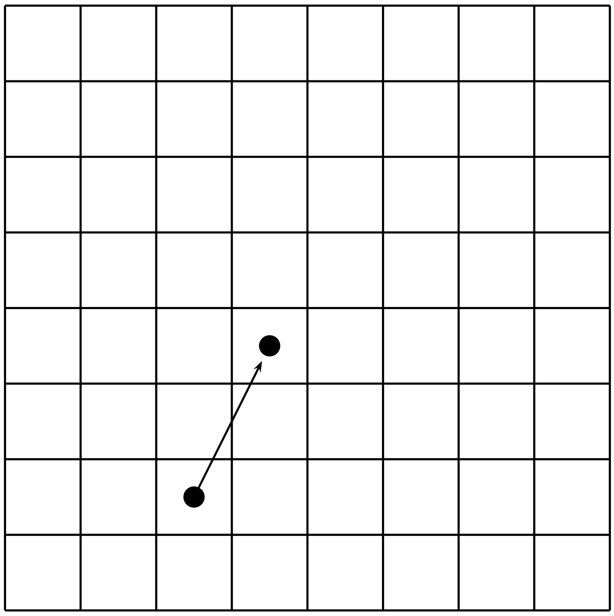
$$\rho'(v_1) + \rho'(v_2) \leqslant n - 1.$$

Yet

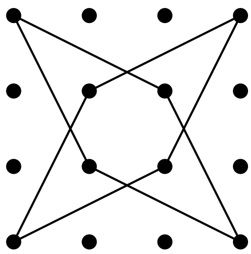
$$\rho'(v_1) + \rho'(v_2) \geqslant n/2 + n/2 = n,$$

and we have a contradiction. Hence Γ is Hamiltonian. $\square$

Example 9.7 (The Knight's Tour) The Knight's Tour Problem is concerned with the use of a chessboard and the piece known as the knight. Can a knight vist each square of a chessboard by a sequence of knight's moves and finish on the same square that it began on?



The solution is to find a link between this problem and the finding of Hamiltonian circuits in a graph. To simplify the explanation, we consider the situation of a 4×4 chessboard (rather than the usual 8×8 one). We represent each square on this chessboard by a vertex of a graph and we join two vertices by an edge if a knight could make a move from between the corresponding squares. The following illustrates some of the edges (more need to be added):



This graph is not Hamiltonian. The corner vertices all have degree precisely two and consequently the eight edges shown would have to be included in any Hamiltonian circuit. This shows that no such Hamiltonian circuit can exist since each of the centre four vertices must be visited at least twice.

It can be shown that there is no Knight's Tour for a chessboard with an odd number of squares (e.g., a 5×5 board). However for some other boards there are solutions (e.g., for 6×6). The solution for the standard 8×8 board was given by Euler in 1759.