Dynamic Programming Theory

*Invented in 1953 by Richard Bellman *1920 1984*

Dynamic programming formulates rules to be used for multistage decision making processes in which the decisions taken previously limit the current choice. The consequences of each decision taken may be assigned numeric values with the aim to maximise or minimise a criterial function related to the entire process in question.

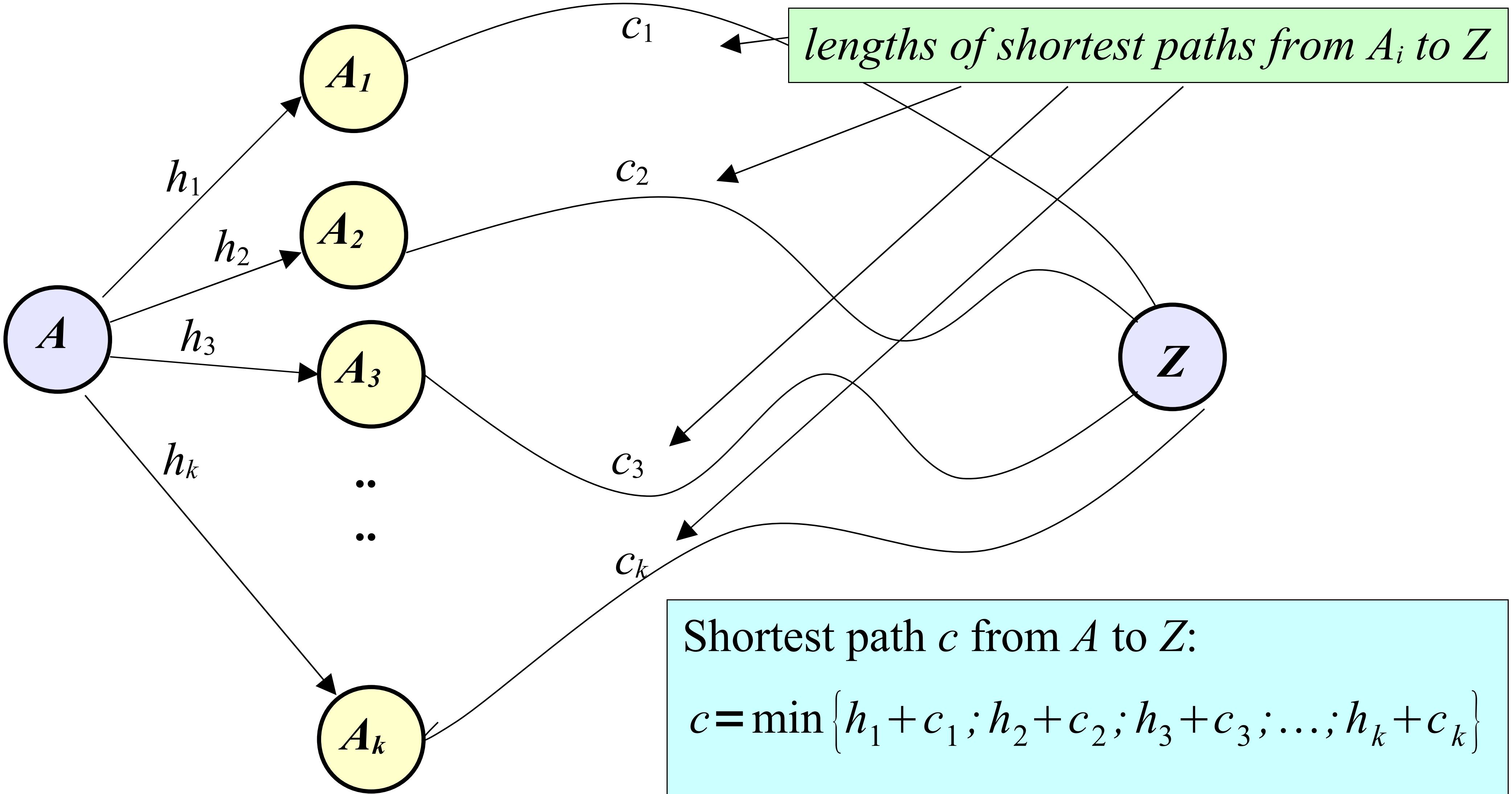
Bellman's Principle of Optimality

Each optimal strategy has the following property:

Whichever the initial state and decision (as part of an optimal strategy) has been, the subsequent decision must itself represent a strategy optimal with respect to the state resulting from the initial decision.

Example

Shortest path from node A to node Z in a directed graph with weighted arcs.

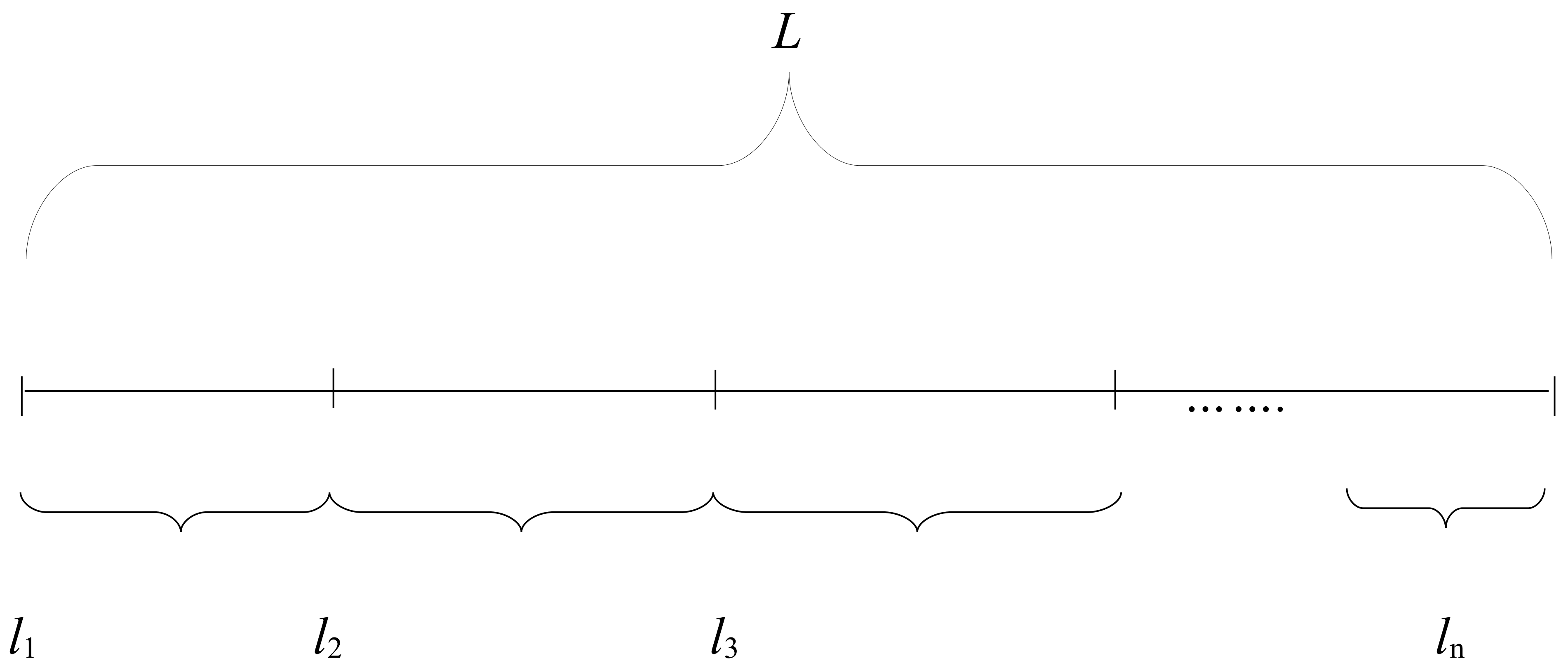


Shortest path c from A to Z:

$$c = \min\{h_1 + c_1; h_2 + c_2; h_3 + c_3; \ldots; h_k + c_k\}$$

Example

Cut a length L into n pieces for the product P of their lengths l_i to be maximal.



$$P = \prod_{i=1}^{n} l_i$$

Denote by $f(x, t)$ the maximum product of x lengths cut from total length t. If, initially, we decide to cut off length y, we will have to find a solution to the problem $f(x-1, t-y)$. If the initial decision has been "the right one", by the optimality principle, $y \cdot f(x-1, t-y)$ will be a solution to the problem.

Clearly, $f(1, t) = t$, so $f(2, t) = \max\limits_{0 \le y \le t} y \cdot (t-y)$.

Here, $y = \frac{t}{2}$ may be chosen since it is the maximum of a function of one variable. Thus $f(2,t) = \frac{t^2}{4}$. Similarly, $f(3,t) = \max_{0 \le y \le t} y \cdot \frac{(t-y)^2}{4}$ so we have a maximum at $y = \frac{t}{3}$. In this way, we can prove that, in an optimal solution, the lengths of all the parts must be the same.

DP problems may be classified according to the following aspects

- How many different decisions must be made
- How many stages the problem involves
- Are the situations involved deterministic or stochastic
- What types of criterial and decision functions are used

- The number of different decisions to be made at each stage can be finite (shortest path) or infinite (cutting a length).
- Some problems may involve an infinite number of stages. A buyer, for example, has a choice of ordering a product or not at any given moment.
- With deterministic situations, the result of a decision is known before it is taken. With stochatic situations, the probability of any possible outcome is known rather than the result of a decision. The function to be optmised in this case represents some kind of average.
- A criterial function depends on the decision and the number of stages. A decision function specifies the decision to be taken to arrive at an optimal solution. In the last example, y indicated the decision while t^n/n^n was the criterial function (the product value after all the $n - 1$ decisions)

A certain type of problems that occur very often

A resource C is available and should be allotted to n activities to reach the maximum profit. The revenue an activity brings depends on the amount of resource allotted to it. Revenues from different activities are independent and additive.

This can be mathematically formulated as follows:

Maximise $\sum_{i=1}^{n} f_i(x_i)$ subject to the constraint $\sum_{i=1}^{n} x_i \leq C$.

This problem is often faced by companies if a number of investments are to be made.

Let $R_k(x)$ be the maximal revenue attainable by dividing a total resource x into the first k activities. Clearly, $R_1(x) = f_1(x)$ where $f_1(x)$ is an increasing function. $R_2(x)$ is the maximal revenue from the first two activities. Adding amount x_2 of the resource to the second activity and the rest to the first one, we will have a total revenue of $f_1(x - x_2) + f_2(x_2)$, which is to be maximised subject to $0 \le x_2 \le x$ so that $R_2(x) = \max\limits_{0 \le x_2 \le x} \{f_2(x_2) + R_1(x - x_2)\}$.

Proceding further by the principle of optimality, we finally obtain:

$$R_n(x) = \max_{0 \le x_n \le x} \{f_n(x_n) + R_{n-1}(x - x_n)\}$$

This iterative procedure can be used to find the optimal strategy provided that an analytic or numeric method is given to calculate maxima or minima of the functions $f_1(x_1)$.

Here, the values x_2, x_3, … x_n are the decision functions while $R_n(C)$ is the criterial function.

Example

Suppose that 100 \$ are to be spent on six subsequent activities to maximise the total revenue. The revenues $f_i(x)$ from the six activities are given as:

$\sqrt{x}$, $2\sqrt{x}$, $3\sqrt{x}$, $4\sqrt{x}$, $5\sqrt{x}$, $6\sqrt{x}$.

Find the value of the maximum total revenue.

$$R_1(100)=\sqrt{100}=10, \quad R_1(100-y)=\sqrt{100-y}$$

$R_2(x)=\max\limits_{0\le y\le 100}\left\{\sqrt{x-y}+2\sqrt{y}\right\}$ where y is the amount spent on the second activity. The maximum is reached for $y=\dfrac{4}{5}x$ so that $R_2(x)=\sqrt{5x}$

Similarly $R_3(x) = \max_{0 \le y \le 100} \left\{ \sqrt{5(x-y)} + 3\sqrt{y} \right\}$ with a maximum at $y = \frac{9}{14}x$, so that $R_3(x) = \sqrt{14x}$.

In a similar way, we can calculate $R_4(x) = \sqrt{30x}$, $R_5(x) = \sqrt{55x}$, and $R_6(x) = \sqrt{91x}$. This means that $R_6(100) = 10\sqrt{91} = 953.94$ and the maximum revenue is 953.94 \$.

The amounts to be spent can then be calculated as follows:

$x_6 = 39.56$ \$, $x_5 = 27.47$ \$, $x_4 = 17.58$ \$

$x_3 = 9.89$ \$, $x_2 = 4.40$ \$, $x_1 = 1.10$ \$