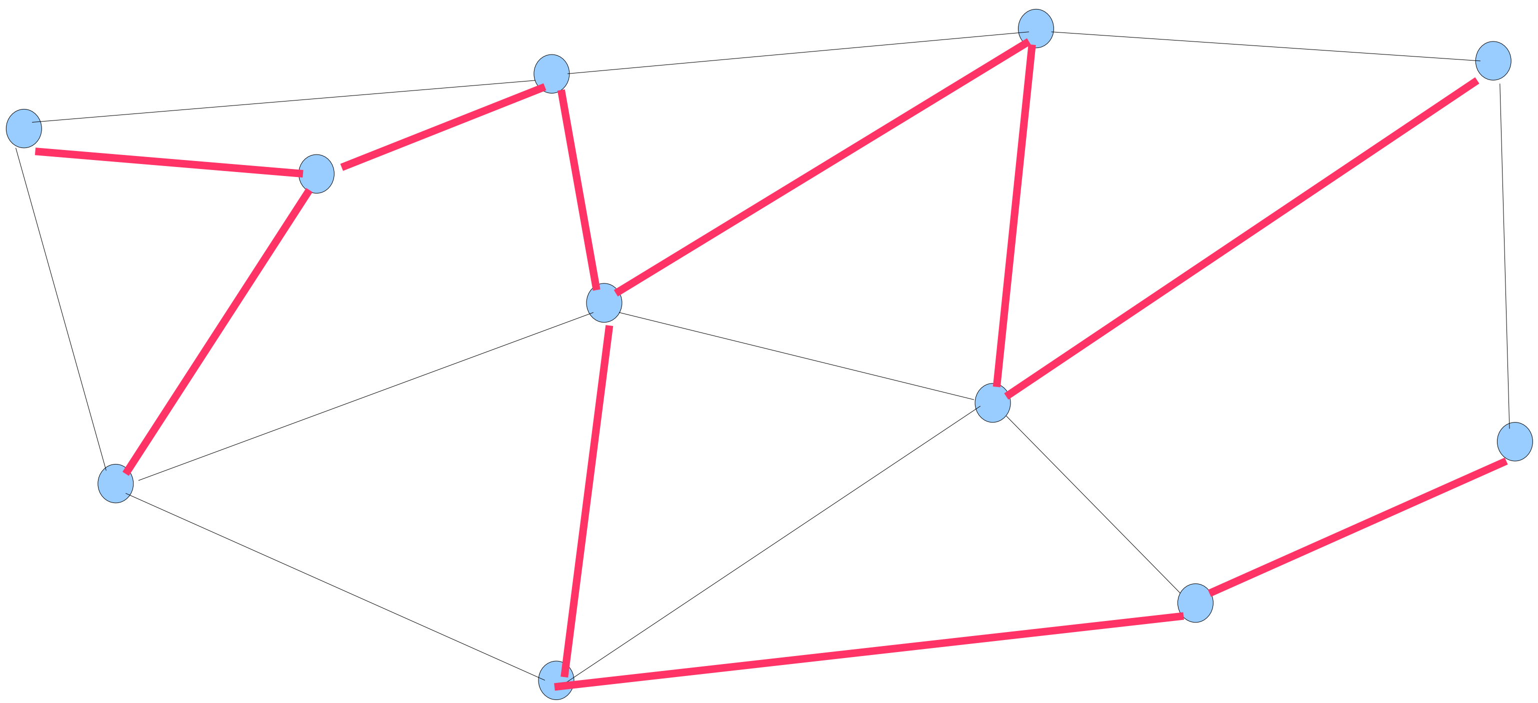
Spanning tree

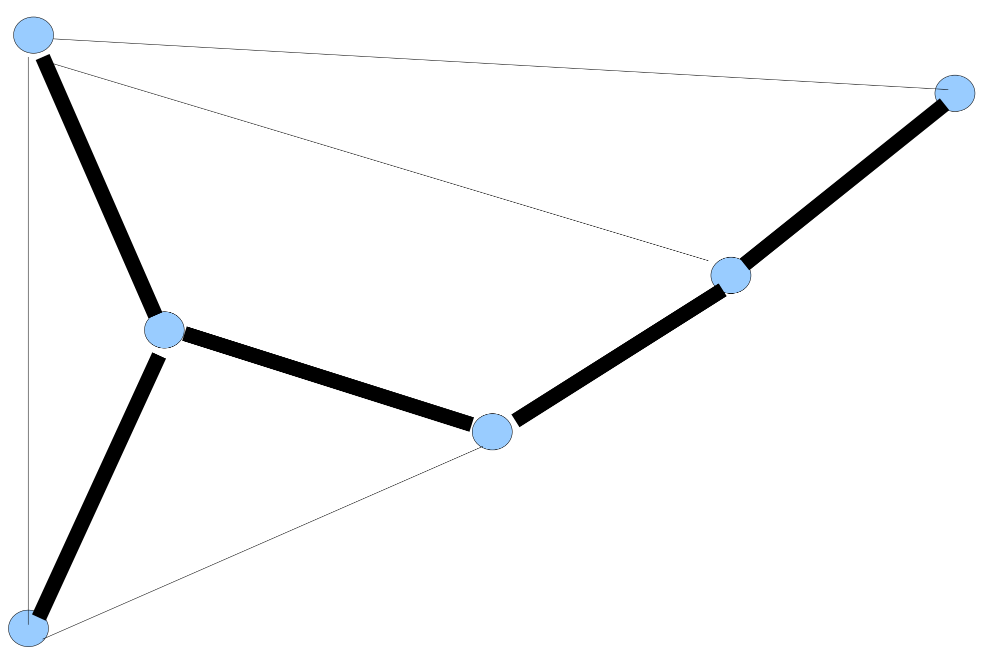
Given a simple graph $G=(N,E)$, its subgraph $S=(N,E')$ is called a ***spanning tree*** of G, if S is a tree.

Each spanning tree of G is thus a tree with the same set of nodes as G.



Chords

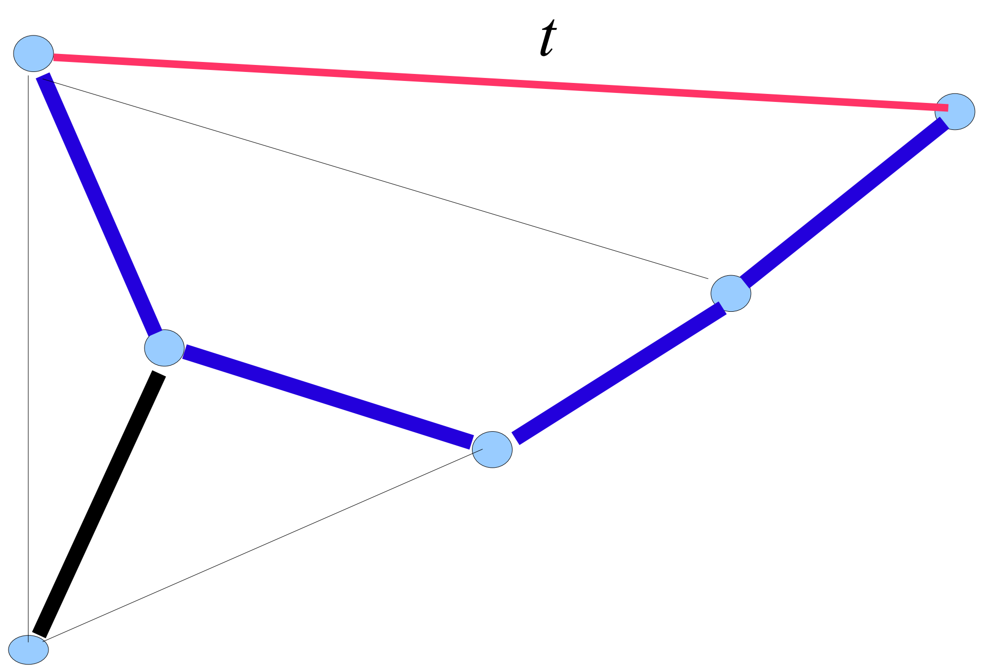
Let $G=(N,E)$ be a simple graph and $S=(N,E')$ its spanning tree. Then, clearly, $E'\subseteq E$ and the edges in E that are not in E' are called ***chords*** of the spanning tree S.



In the above picture, the thin edges are the chords

Fundamental circles

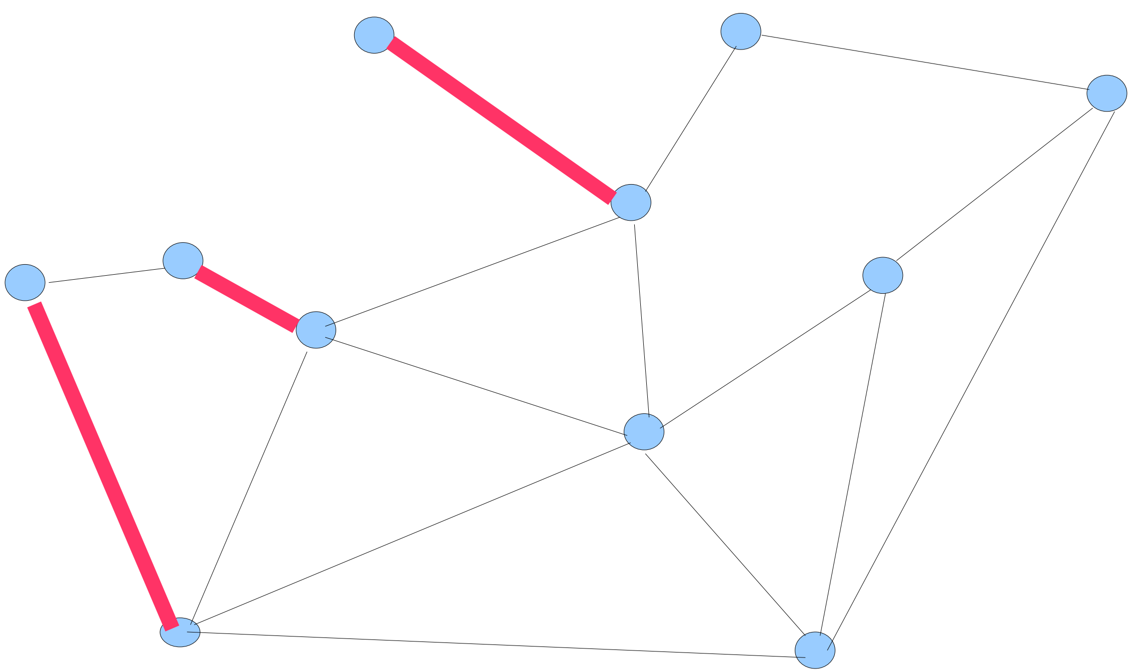
Let $G=(N,E)$ be a simple graph, $S=(N,E')$ its spanning tree and let $t=(u,v)$ be a chord of S. Then there is a unique path between nodes u and v in S. Combined with the chord t, this unique path then clearly forms a circle $C^S(t)$ in G called a ***fundamental circle*** of S with respect to t



The red chord and blue edges form a fundamental circle of the spanning tree denoted by bold edges.

Disconnecting set

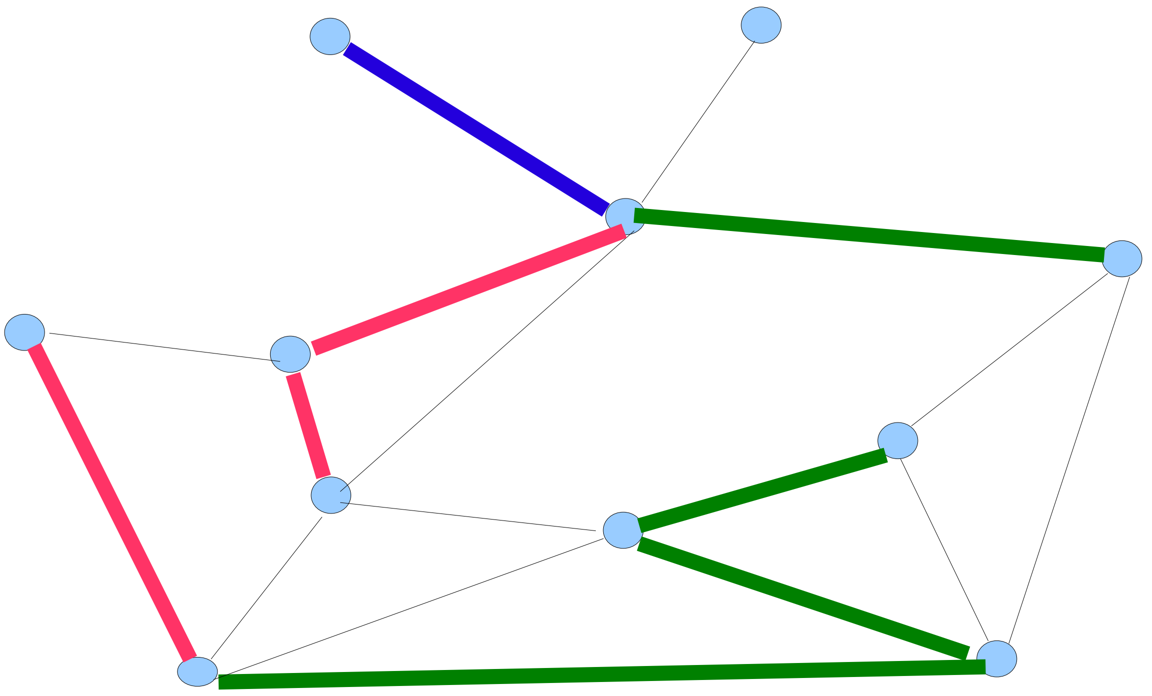
Let $G=(N, E)$ be a simple connected graph and $D \subseteq E$. The subset D is called a ***disconnecting set*** of G if G becomes disconnected by removing from G all edges in D. Thus $G'=(N, E-D)$ is not connected.



Red edges form a disconnecting set in the picture above

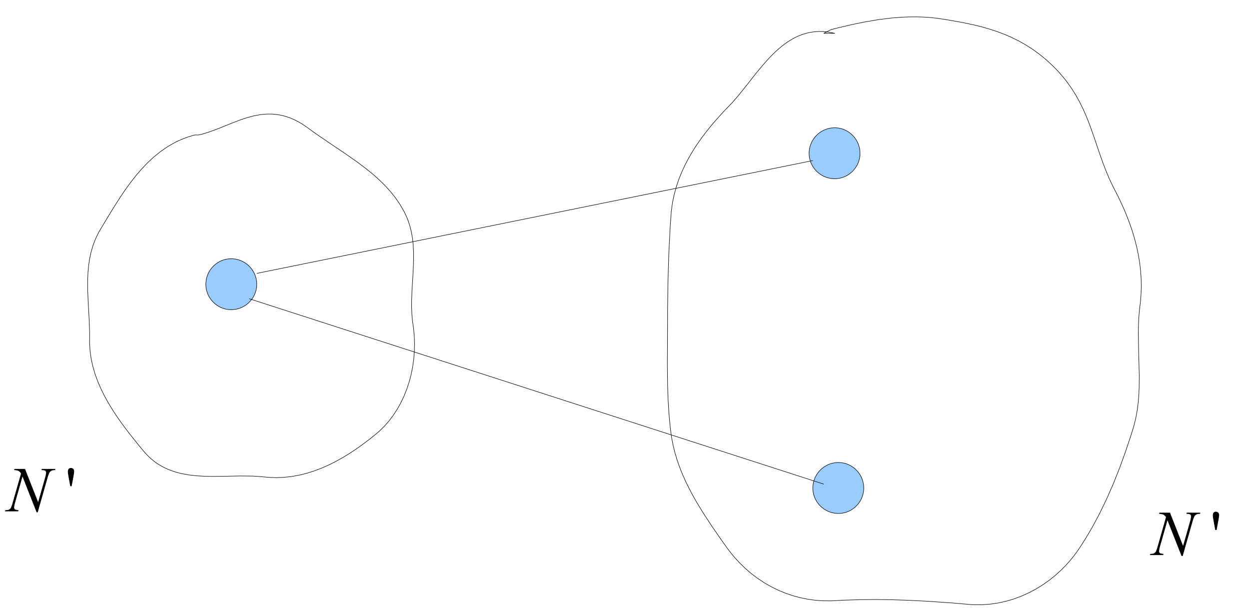
Cut

Let $G=(N,E)$ be a simple graph and $D \subseteq E$. We call D a cut of G if it is the minimum disconnecting set of *G,* that is, if no proper subset of D is a disconnected set of G.



Edges of the same colour are cuts of the above graph

If D is a cut in a connected graph $G=(N, E)$, then removing from G all edges in D clearly splits G into two components $G'=(N', E')$ and $G''=(N'', E'')$. On the other hand, if N is split into two parts N' and N'', the disconnecting set $D=\{e=(u, v) | e \in E \wedge u \in N' \wedge v \in N''\}$ is not necessarily a cut as seen in the picture below. The question is what are the conditions that necessitate this. The answer is given by the following theorem.



Theorem 3

Let $G=(N,E)$ be a simple connected graph and let $N=N'\cup N''$ with $N'\cap N''=\emptyset$. Then the set $D=\{e=(u,v)\mid e\in E\wedge u\in N'\wedge v\in N''\}$ is a cut of G, if the following conditions are fulfilled :

(a) for any $u_{1,}u_p\in N'$ there is a path $(u_1,e_1,u_2,\ldots,u_{p-1},e_{p-1},u_p)$ with $u_i\in N', 1\le i\le p$

(b) for any $v_{1,}v_q\in N''$ there is a path $(v_1,f_1,v_2,\ldots,v_{q-1},f_{q-1},v_q)$ with $v_i\in N'', 1\le i\le q$

This theorem says that each of the components created by removing from G the edges in the disconnecting set must be connected using only "its own nodes".

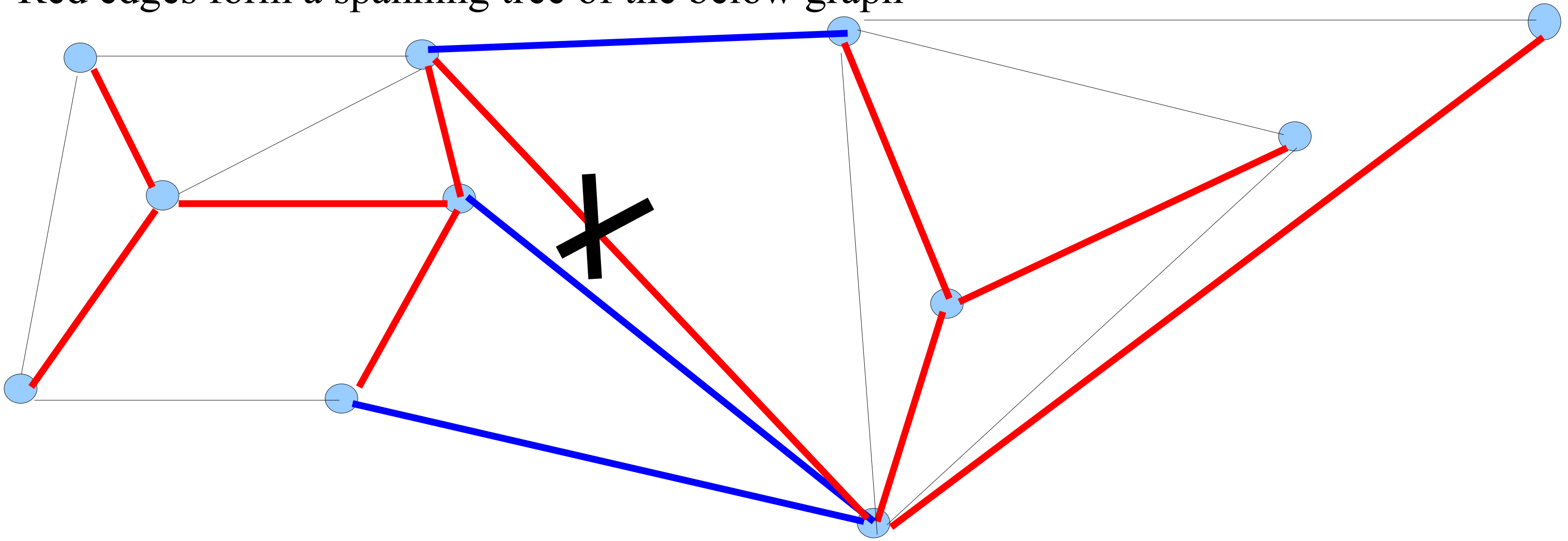
Proof

Suppose that, for a disconnecting set *D*, conditions (a) and (b) are fulfilled and D is not a cut of *G*. Then there is a set $D' \subset D$ which is also disconnecting and so there is an edge $e = (u, v)$ such that $e \in D \wedge e \notin D' \wedge u \in N' \wedge v \in N''$. Let $x \in N'$ and $y \in N''$ be any two nodes in *G*. Then there is a path $(x, e_1, u_{2,} \ldots, u_{p-1}, e_{p-1}, u), x \in N', u_i \in N', 2 \leq i \leq p-1$ and a path $(y, h_1, v_{2,} \ldots, v_{q-1}, h_{p-1}, v), y \in N'', v_i \in N'', 2 \leq i \leq q-1$, which means that there is a path between *x* and *y* that has only the edge *e* in common with *D* and so *D* ' is not disconnecting, which is a contradiction.

Corollary 4

Let $G=(N,E)$ be a simple graph and $S=(N,E')$ its spanning tree. Removing any edge from S makes S disconnected with two components $S_1=(N_1,E_1)$ and $S_2=(N_2,E_2)$ that are trees with $N_1\cap N_2=\emptyset, N_1\cup N_2=N$ and the set $D=\{e=\{u,v\}\,|\,u\in N_1\wedge v\in N_2\wedge e\in E\}$ is a cut of G.

Red edges form a spanning tree of the below graph

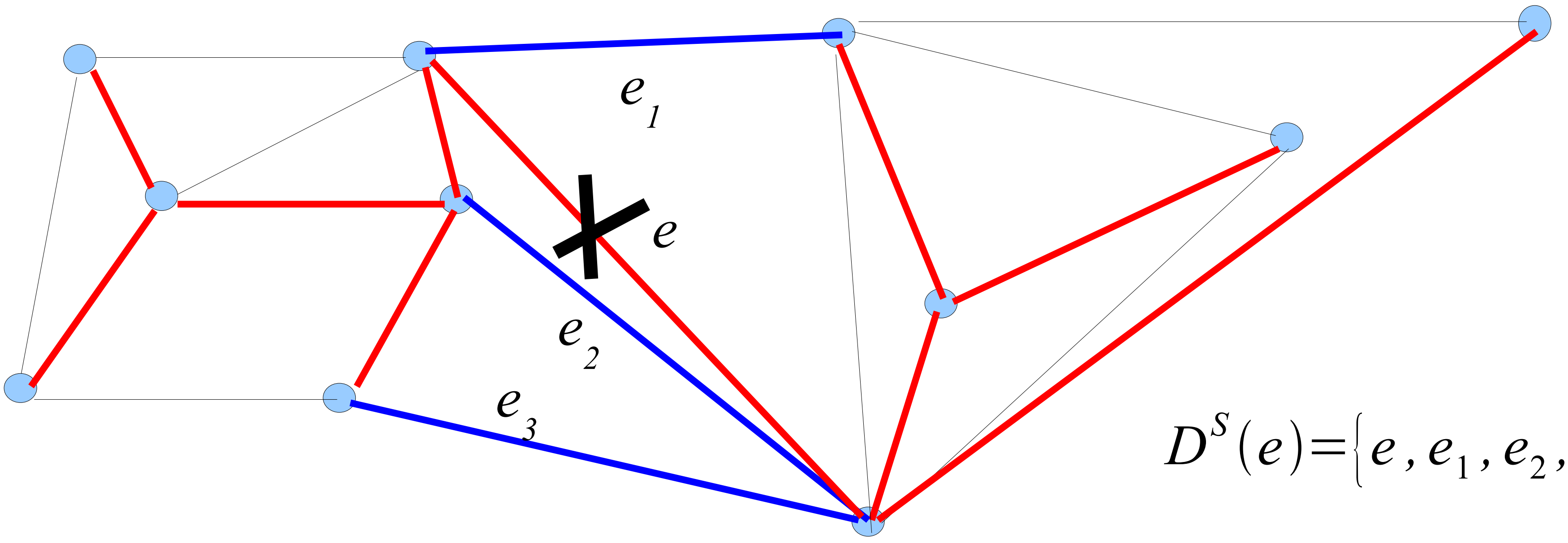


Blue edges together with the removed red one form the resulting cut

Fundamental cut

Let $G=(N,E)$ be a simple graph and $S=(N,E')$ its spanning tree.
Remove an edge e from S to get two trees $S_1=(N_{1,}E_1)$ and $S_2=(N_{2,}E_2)$ and let $D=\{e=\{u,v\}|e\in E\wedge u\in N_1\wedge v\in N_2\}$ be the cut thus created. Then D is called the fundamental cut of S created by the edge e and is denoted by $D^S(e)$.

The red edges form a spanning tree S of the below graph



$$D^S(e)=\{e,e_1,e_2,e_3\}$$

Theorem 5

Let $G=(N,E)$ be a simple graph and $S=(N,E')$ its spanning tree.

Let C be a circle in G and D a cut of G. Then:

(a) C and D have either no or an even number of edges in common,

(b) at least one edge of C is a chord of S,

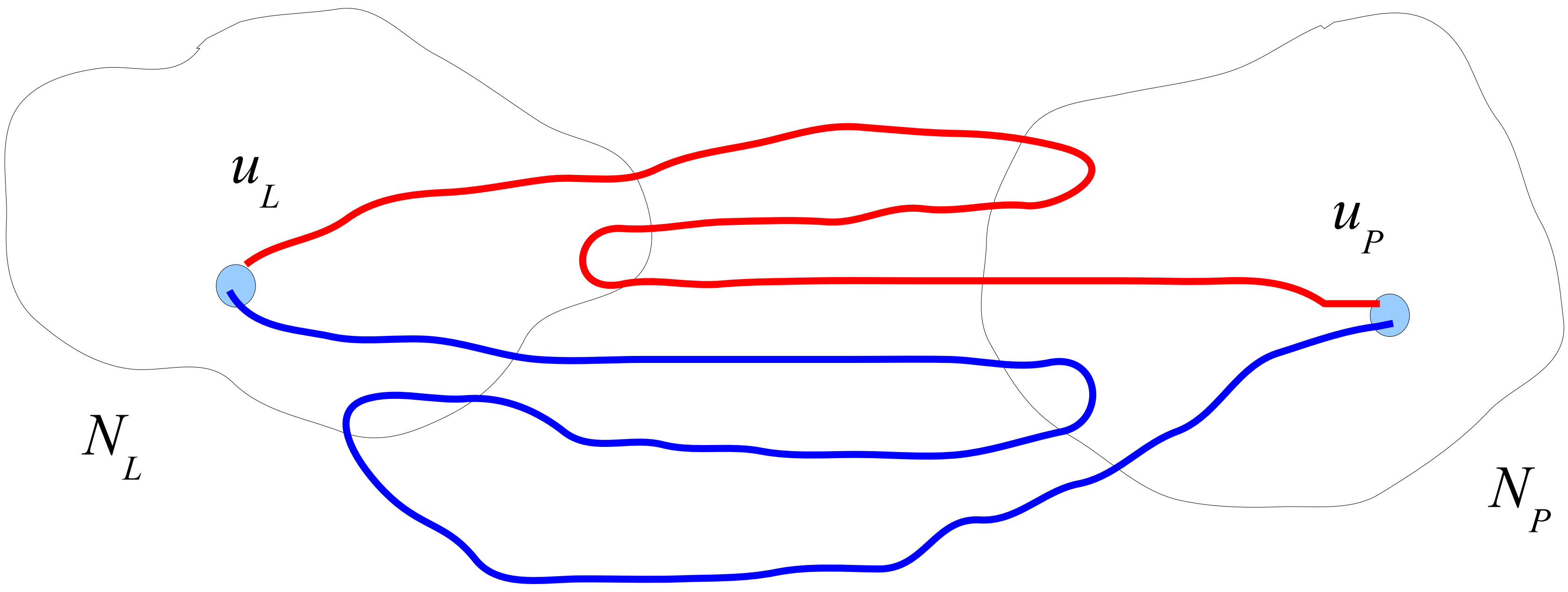
(c) at least one edge of D is in S.

Proof:

(a) Clearly, we can write

$$D=\{e=\{u,v\} \mid e\in E\wedge u\in N_L\wedge v\in N_P, N_L\cup N_P=N, N_L\cap N_P=\emptyset\}$$

If all the nodes of circle C are in one of the subsets N_L, N_P, then of course C and D have no edges in common. Let then u_L and u_P be nodes such that $u_L\in U_L$ and $u_P\in U_P$. Thus each circle containing nodes u_L and u_P clearly defines two paths P_1 and P_2 between u_L and u_P such that their node sets are (except for u_L and u_P) disjunct. As can be seen in the picture below, walks along P_1 and along P_2 both pass the "gap" between N_L and N_P an odd number of times. Each pass is over a different edge in D. Thus the number of edges that C and D have in common is even.



(b) If no edge of C is a chord of S, then the entire C is contained in S, which is a contradiction.

(c) If D and K have no edge in common, D cannot be a disconnecting set as S is a connected graph.

Theorem 6

Let $G=(N,E)$ be a simple graph and $S=(N,E')$ its spanning tree.

(a) Let $D^S(f)$ be the fundamental cut of S created by $f \in H'$ and e another edge in this cut. Then:

(1) e is a chord of S creating a fundamental circle $C^S(e)$

(2) f is contained in the fundamental circle $C^S(e)$

(3) if t is a chord of S, $t \notin D^S(f)$, then $f \notin C^S(t)$

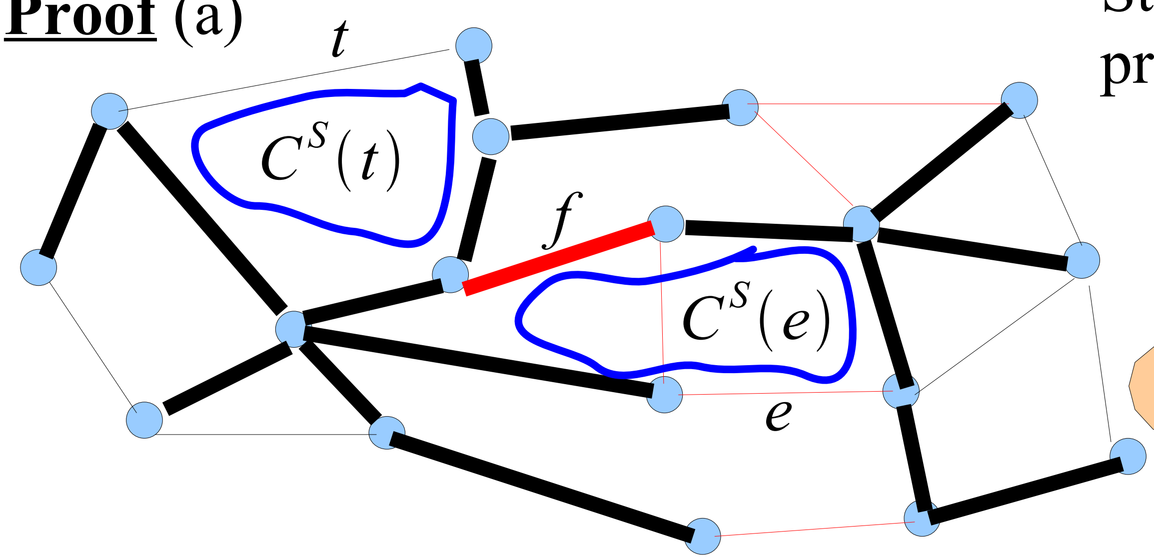
(b) Let $C^S(f)$ be a fundamental circle of S created by f and e another edge in this circle. Then:

(1) e is an edge in S creating a fundamental cut $D^S(e)$

(2) f is contained in the fundamental cut $D^S(e)$

(3) if t is another edge of S, $t \notin C^S(f)$, then $f \notin D^S(t)$

Proof (a)



Statements under (b) can be proved much like those under (a).

Set $D^S(f)$ is formed by red edges

(1) No edge $e \in D^S(f)$ can be contained in S as S is a tree. Thus e is a chord creating a fundamental circle $C^S(e)$

(2) Let $e \in D^s(f), e \neq f$. Then we can write $D^s(f) = \{f, e\} \cup A$ where A is the set of chords of S. Similarly, $C^S(e) = \{e\} \cup B$ where B is the set of edges of S. Obviously $e \in D^s(f) \cap C^S(e)$. By Theorem 5, $|D^S(f) \cap C^S(e)|$ is even. As $A \cap B = \emptyset$, this means that $D^S(f) \cap C^S(e) = \{e, f\}$ Thus f is an edge in the fundamental circle $C^S(e)$.

(3) Let t be a chord of S and suppose that f is contained in circle $C^S(t)$. Then, clearly, the two nodes incident on t will be in different components S_1 and S_2 defined by edge f. However, this means that $t \in D^S(f)$, which proves the implication.