NETWORK

A ***network*** is a quadruple $N=(G,s,t,c)$ where $G=(V,A)$ is a simple digraph. Every ordered pair (u,v) of nodes is assigned a non-negative ***capacity*** $c(u,v)\geq 0$. For an ordered pair $(u,v)\notin A$, we put $c(u,v)=0$. There are two special nodes: the ***source*** s and the ***target*** t. In what follows we will assume that each node lies on a directed path from the source to the target.

A ***flow*** in a network $N=(G,s,t,c)$ is a mapping $f:V\times V\rightarrow R$ satisfying the following three conditions:

1. $f(u,v)\leq c(u,v), \forall u,v\in V$
2. $f(u,v)=-f(v,u), \forall u,v\in V$
3. $\sum_{v\in V} f(u,v)=0, \forall u\in V-\{s,t\}$

The quantity $f(u,v)$, which may be positive, zero, or negative, is called the ***flow from node*** u ***to node*** v.

The quantity $|f|=\sum_{u\in V} f(s,u)$ is called the ***total flow*** of N.

Notes to the definition of flow:

By Condition 2 above, we have $f(u,u)=0$, that is, the flow from a node into itself is zero.

By Condition 3, the total flow from each node different from source and target is zero. Using Condition 2, this can be rewritten as

$\sum_{u \in V} f(u,v)=0, \forall v \in V-\{s,t\}$, that is, the total flow into each node different from source and target is zero.

If no arcs exist between nodes *u* and *v*, there can be no flow between them as $c(u,v)=c(v,u)=0$ and so $f(u,v) \le 0$ and $f(v,u) \le 0$. Condition 2 then yields $f(u,v)=f(v,u)=0$.

Summation formalism

We will use the following short-cut symbol where X and Y are sets of nodes:

$$f(X,Y)=\sum_{x\in X}\sum_{y\in Y} f(x,y)$$

We may also leave out the brackets "{", "}" denoting sets. For example, in the formula $f(s,V-s)=f(s,V)$, $V-s$ is a shortcut for $V-\{s\}$.

Lemma 1

Let $N=(G,s,t,c)$ be a network where $G=(V,A)$ is a simple digraph, $X,Y,Z\subseteq V$ and let f be a flow in N. We have

1. $f(X,X)=0$
2. $f(X,Y)=-f(Y,X)$
3. $f(X\cup Y,Z)=f(X,Z)+f(Y,Z)$ and

   $f(Z,X\cup Y)=f(Z,X)+f(Z,Y)$ if $X\cap Y=\emptyset$

Ford-Fulkerson Method

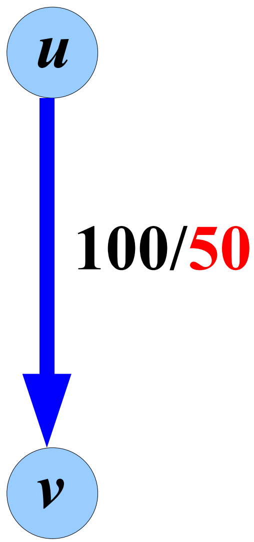
This method rests on the following three basic concepts: ***residual network***, ***augmenting path***, and ***cut***.

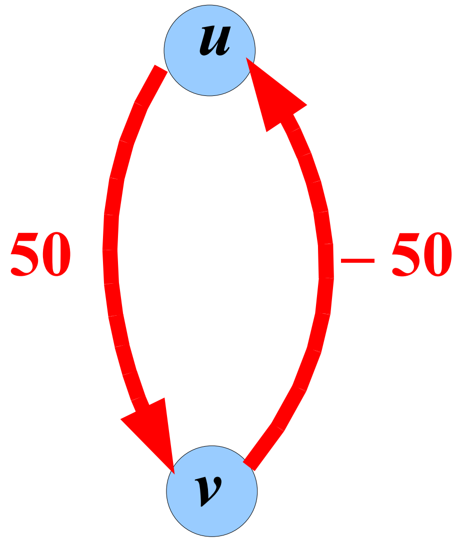
It starts with a zero flow. By each iteration, an augmenting path is found from source *s* to target *t* along which an additional piece of flow can be added to augment the current flow. This process is repeated until no augmenting path can be found.

By establishing a relationship between a maximal flow and a minimal cut, it can be proved that the resulting augmented path is indeed maximal. The basic concepts will now be defined and explained.

Residual capacity

Define a flow *f* in a network $N=(G,s,t,c)$ where $G=(V,A)$ is a simple graph. For every pair of nodes $(u,v), u,v \in V$, define the ***residual capacity*** as $c_f(u,v)=c(u,v)-f(u,v)$.





Residual network

Let $N=(G,s,t,c)$ be a network with $G=(V,A)$ being a simple digraph. Let f be a flow in N. Let c_f be the residual capacity of N defined by the flow f. Define a simple digraph $G_f(V,A_f)$ where $A_f=\{(u,v)|u,v\in V \wedge c_f(u,v)>0\}$. The network $N_f=(G_f,s,t,c_f)$ is called the ***residual network*** of network N with respect to flow f.

Lemma 2

Let $N=(G,s,t,c)$ be a network and *f* a flow in *N*. Let $N_f=(G_f,s,t,c_f)$ be the residual network of *N* with respect to *f* and let f^* be a flow in N_f. Then the mapping

$$(f+f^*):V\times V\rightarrow R$$

defined as

$$(f+f^*)(u,v)=f(u,v)+f^*(u,v)$$

is a flow in network *N*.

Augmenting path

Let $N=(G,s,t,c)$ be a network. Define a flow f in N and the residual capacity c_f with respect to this flow. Using c_f, a simple digraph $G_f(V,A_f)$ may be built as above. Let now P be any path in G_f from source s to target t in. Such a path is called an ***augmenting path*** in N with respect to flow f. Let $P=(s=v_{0,} v_{1,} v_{2,} \ldots, v_{k-1}, v_k=t)$. We define the ***residual capacity*** $c_f(P)$ of P with respect to f as follows:

$$c_f(P)=\min\{c_f(v_i,v_{i+1})|i=0,1,\ldots,k-1\}$$

Lemma 3

Let $N=(G,s,t,c)$ be a network, define a flow f in N and the residual network $N_f=(G_f,s,t,c_f)$. Let $P=(s=v_{0,} v_{1,} v_{2,} \ldots, v_{k-1}, v_k=t)$ be an augmenting path in N_f and $c_f(P)$ its residual capacity with respect to f.

Define a mapping $f_P: V \times V \rightarrow R$ as follows:

1. $f_P(v_i, v_{i+1}) = c_f(P), i=0,1,2,\ldots,k-1$
2. $f_P(v_{i+1}, v_i) = -c_f(P), i=0,1,2,\ldots,k-1$
3. $f_P(u,v)=0$ otherwise

Then f_P is a flow in G_f with $|f_P| = c_f(P) > 0$

Corollary 4

Let $N = (G, s, t, c)$ be a network, f a flow defined in *N*, $N_f = (G_f, s, t, c_f)$ the residual network with respect to *f*, and *P* an augmenting path in G_f with $c_f(P)$ as its residual capacity. Let a mapping $f_P : V \times V \rightarrow R$ be defined as in Lemma 3. Define a mapping $f^* = f + f_P : V \times V \rightarrow R$ as in Lemma 2. Then f^* is a flow in *N* with $|f^*| = |f| + |f_P| > |f|$.

Network cut and capacity

Let $N=(G,s,t,c)$ be a network with $G=(V,A)$ being a simple digraph. We will call any partition (S,T) of V a ***cut*** of $N=(G,s,t,c)$ if $s\in S$ and $t\in T$. Recall that, for a partition (S,T) of V, we have $S\cup T=V$ and $S\cap T=\emptyset$.

Given a flow f in N, we define the ***flow over cut*** (S,T) with respect to f as $f(S,T)$ and the ***capacity of cut*** (S,T) as $c(S,T)$.

Note that, when computing the flow over a cut, the sum may include negative flows between nodes whereas the capacity of a cut is always composed of non-negative values.

<u>Lemma 5</u>

Let $N=(G,s,t,c)$ be a network, f a flow in *N*, and let (S,T) be a cut in *N*. Then the flow over the cut (S,T) with respect to *f* is equal to the value of the flow *f*, that is, $f(S,T)=|f|$.

Proof:

Since $S\cup T=V$ and $S\cap T=\emptyset$, we have $T=V-S$ and, using Lemma 1, we can write $f(S,T)=f(S,V)-f(S,S)=f(S,V)=$

$$=f(s,V)+f(S-s,V)=f(s,V)=|f| .$$

An immediate consequence of the above lemma is that the value of a flow is equal to the total flow into the target.

The next corollary shows how capacities can be used to establish an upper bound for a flow.

<u>Corollary 6</u>

The value of a flow in a network is less than the capacity of any cut of this network.

<u>Proof:</u> Let (S,T) be a cut. By Lemma 5 and by condition 1 in the definition of a flow, we have

$$|f| = f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) \le \sum_{u\in S}\sum_{v\in T} c(u,v) = c(S,T) .$$

<u>Theorem 7</u> (on maximal flow and minimal cut)

Let $N=(G,s,t,c)$ be a network and f a flow in N. Then the following conditions are equivalent:

(a) f is a maximal flow in N

(b) the residual network $N_f=(G_f,s,t,c)$ contains no augmenting paths

(c) $|f|=c(S,T)$ for a cut (S,T) of N

<u>Proof:</u>

(a)$\Rightarrow$(b) Let f be a maximal flow in N and suppose that there is an augmenting path P in G_f. Then, by Corollary 4, $f^*=f+f_P: V\times V\rightarrow R$ is a flow in N such that $|f^*|>|f|$ which is a contradiction.

$(b) \Rightarrow (c)$: Let there be no augmenting paths in N so that no path exists in G_f from s to t. Define $S = \{u \in V \mid \text{there is a path in } G_f \text{ from } s \text{ to } u\}$, $T = V - S$. Now (S, T) is clearly a partition and a cut since, obviously, $s \in S$ and $t \notin S$ as no path exists in G_f from s to t.

For every $u \in S, v \in T$, we have $f(u, v) = c(u, v)$ since, otherwise, $(u, v) \in A_f$ and so $v \in S$. Therefore, by Lemma 5, $|f| = f(S, T) = c(S, T)$.

$(c) \Rightarrow (a)$: Let $|f| = c(S, T)$ for a cut (S, T) of N and suppose that f is not maximal. This means that there exists a flow f_1 such that $|f_1| > |f|$. However, by Corollary 6, we have $|f_1| \leq c(S, T)$, which is a contradiction.

Ford-Fulkerson algorithm

For a network $N=(G,s,t,c)$ where $G=(V,A)$ is a simple digraph:

1. INITIALIZE: For every $u,v\in V$ put $f[u,v]:=0$

2. BUILD RESIDUAL NETWORK $N_f=(G_f,s,t,c_f)$: For every $u,v\in V$ calculate the residual capacity $c_f[u,v]:=c[u,v]-f[u,v]$ to build the residual graph $G_f(V,A_f)$ with $A_f=\{(u,v)|u,v\in V\wedge c_f(u,v)>0\}$

3. DETERMINE THE EXISTENCE OF A PATH IN G_f BETWEEN *s* AND *t*:

If no path exists: Stop. The current total flow *f* is maximal,

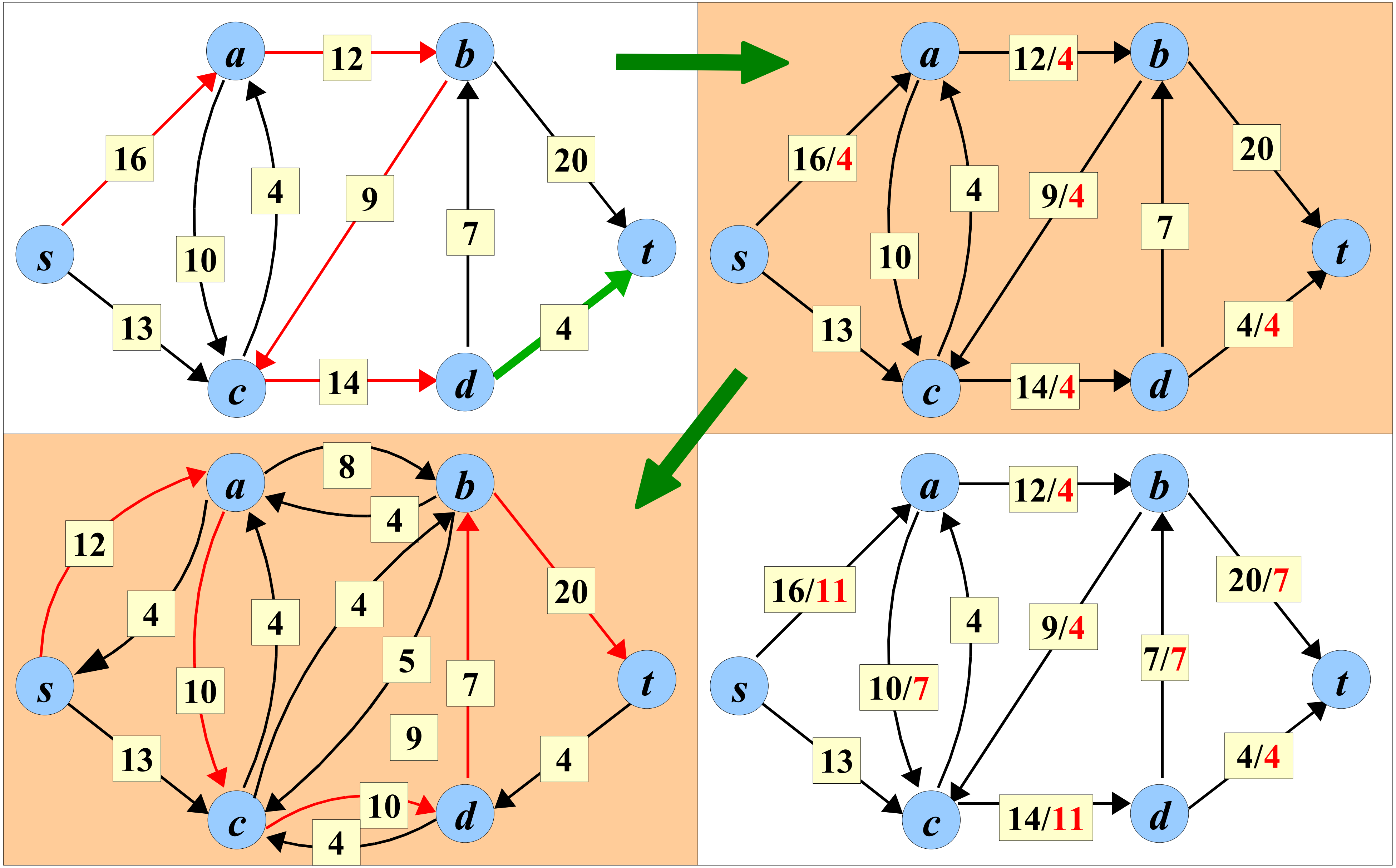
else create path $P=(s=v_{0,}v_{1,}a_{2,}v_{2,}\ldots,v_{k-1},a_k,v_k=t)$ and continue.

4. AUGMENT THE CURRENT FLOW

Calculate the residual capacity of *P*: $c_f(P)=\min\{c_f(v_i,v_{i+1})|i=0,1,\ldots,k-1\}$

Put: $f[v_i,v_{i+1}]:=f[v_i,v_{i+1}]+c_f(P), f[v_{i+1},v_i]:=f[v_{i+1},v_i]-c_f(P), i=1,2,\ldots,k-1$

Go to Step 2.



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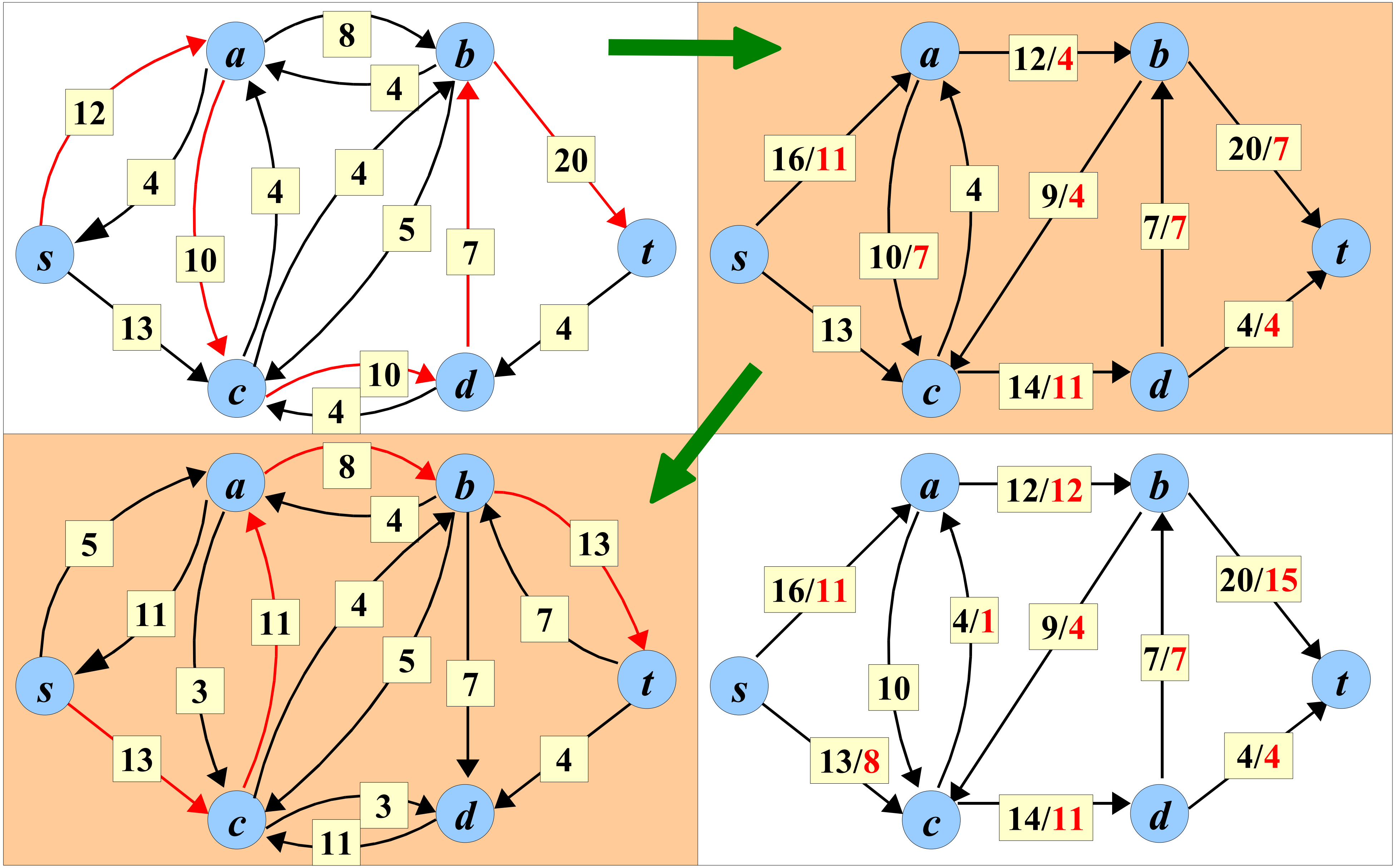

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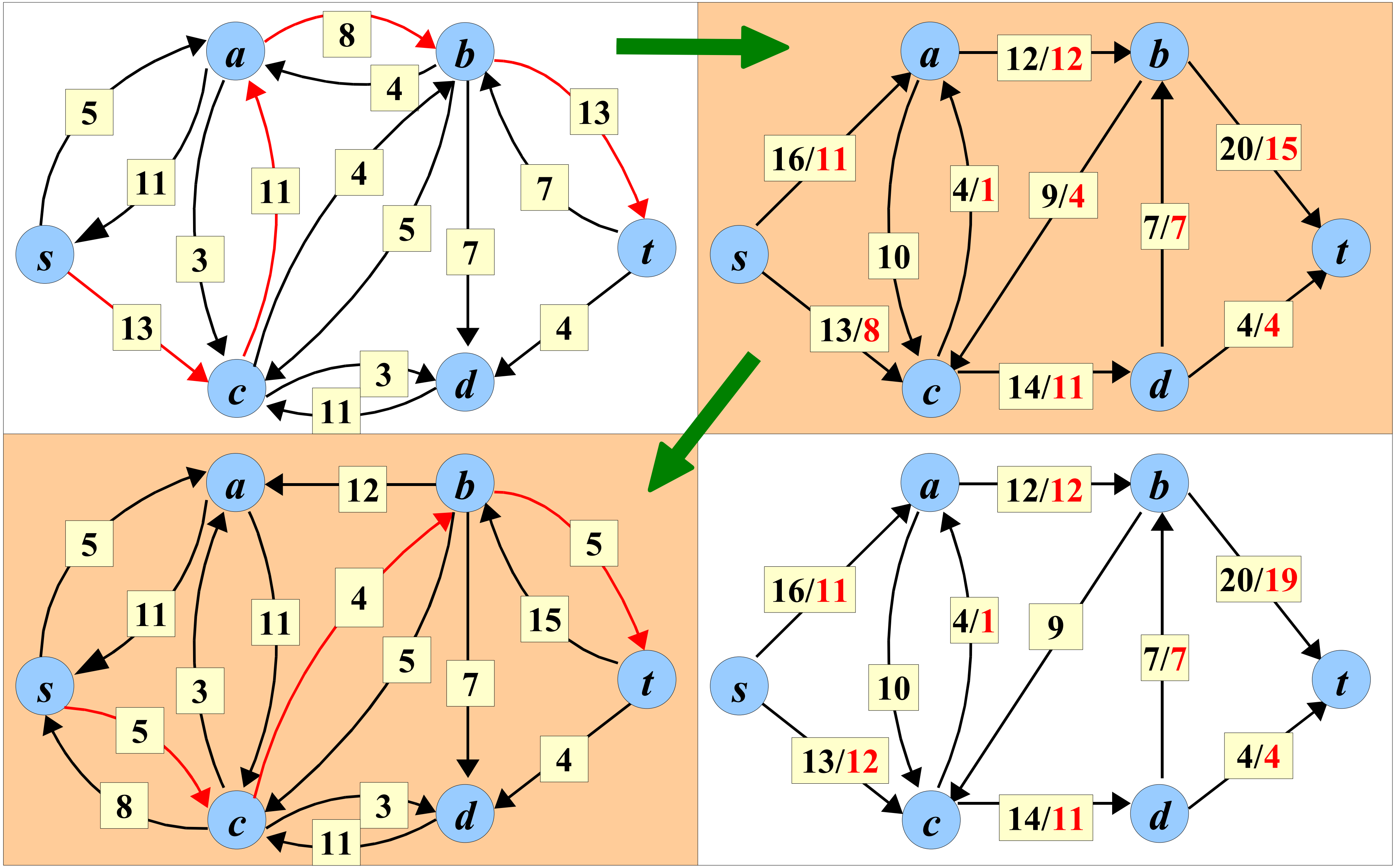

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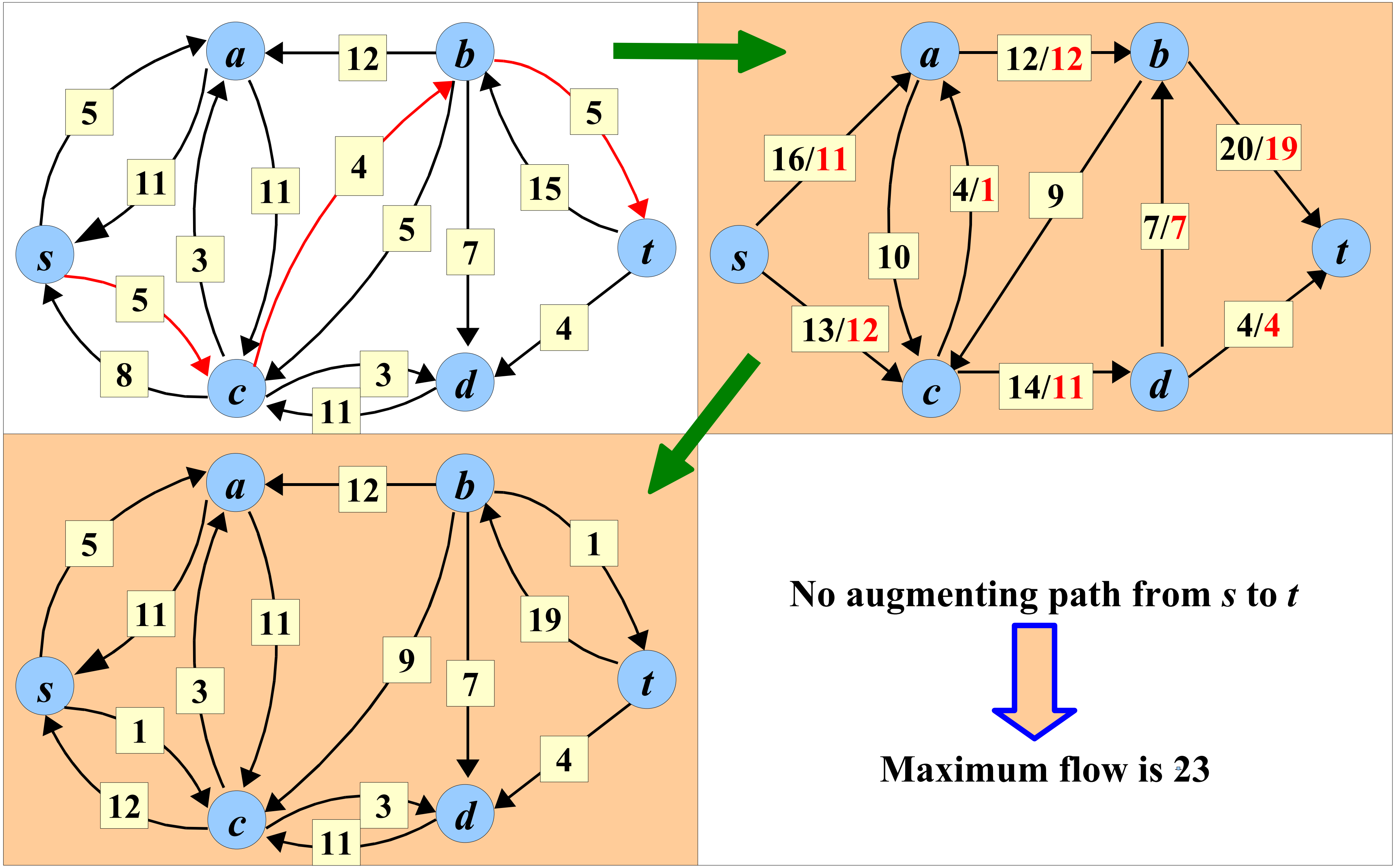

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