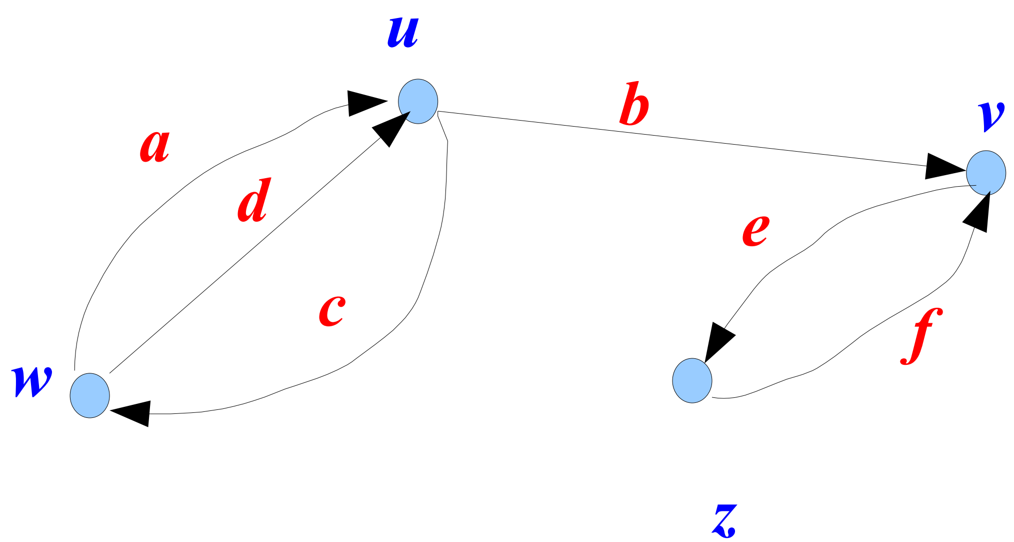
Directed graph

A directed graph is defined as a triple $G=(N, A, \epsilon)$ where N is a finite set of nodes (vertices) and A is a finite set of arcs. The mapping $\epsilon: A \rightarrow \{(u, v) | u, v \in N\}$ assigns an ordered pair (u,v) of nodes to every arc a. We say that arc a leads from node u to node v.

$$\epsilon(a)=\epsilon(c)=(w, u) \quad \epsilon(b)=\epsilon(c)=(u, v)$$

$$\epsilon(d)=(u, w) \quad \epsilon(b)=(u, v)$$

$$\epsilon(e)=(v, z) \quad \epsilon(f)=(z, v)$$

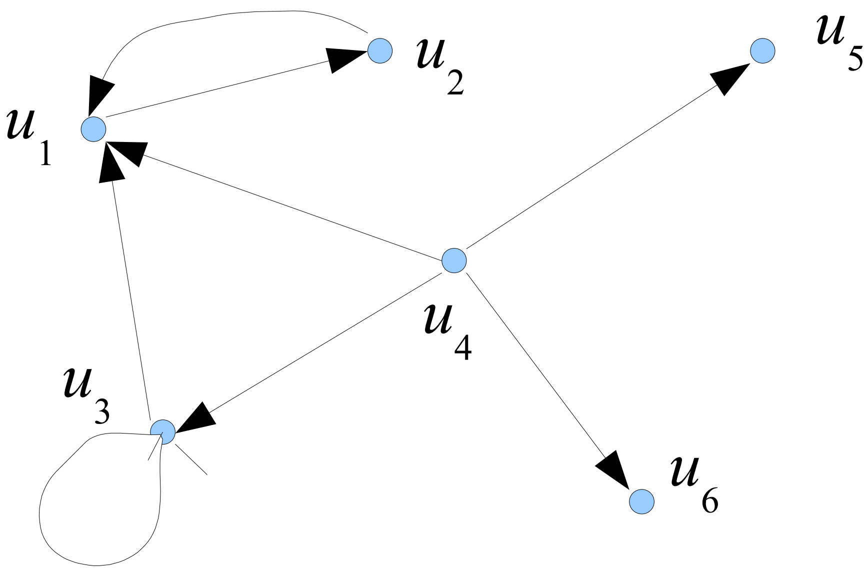


Indegree and outdegree

Let $G=(N, A, \epsilon)$ be a directed graph. For a node $u \in N$ of G define numbers $\deg_+(u)=|M_+|, \deg_-(u)=|M_-|$ with

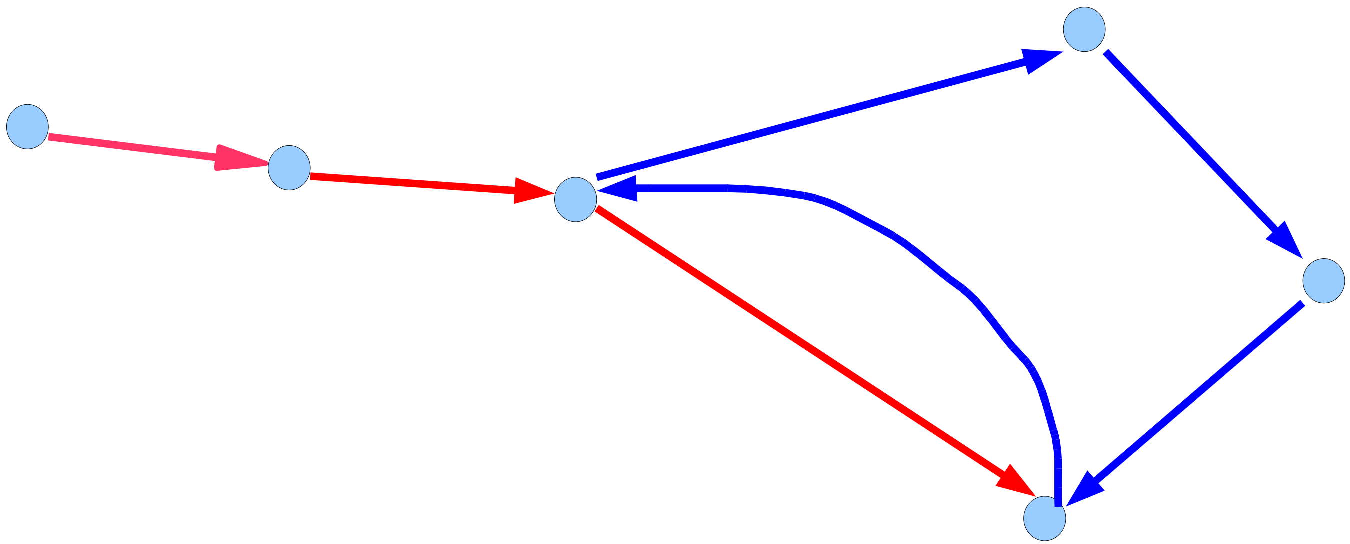
$M_+=\{a \in A \mid \exists v \in N : \epsilon(a)=(v, u)\}$ and $M_-=\{a \in A \mid \exists v \in N : \epsilon(a)=(u, v)\}$

Called **indegree**, $\deg_+(u)$ equals the number of arcs that lead from a node to u. Similarly, $\deg_-(u)$ is called an **outdegree** equaling the number of arcs that lead from u to another node. If $\deg_-(u)=0$, u is called an **end node** and, if $\deg_+(u)=0$, u is called an **initial node** of G.



Node	$\deg_+$	$\deg_-$	
u_1	3	1	
u_2	1	1	
u_3	2	2	loop increases both $\deg_+$ and $\deg_-$
u_4	0	4	initial node
u_5	1	0	end node
u_6	1	0	end node
u_7	0	0	both end and initial node

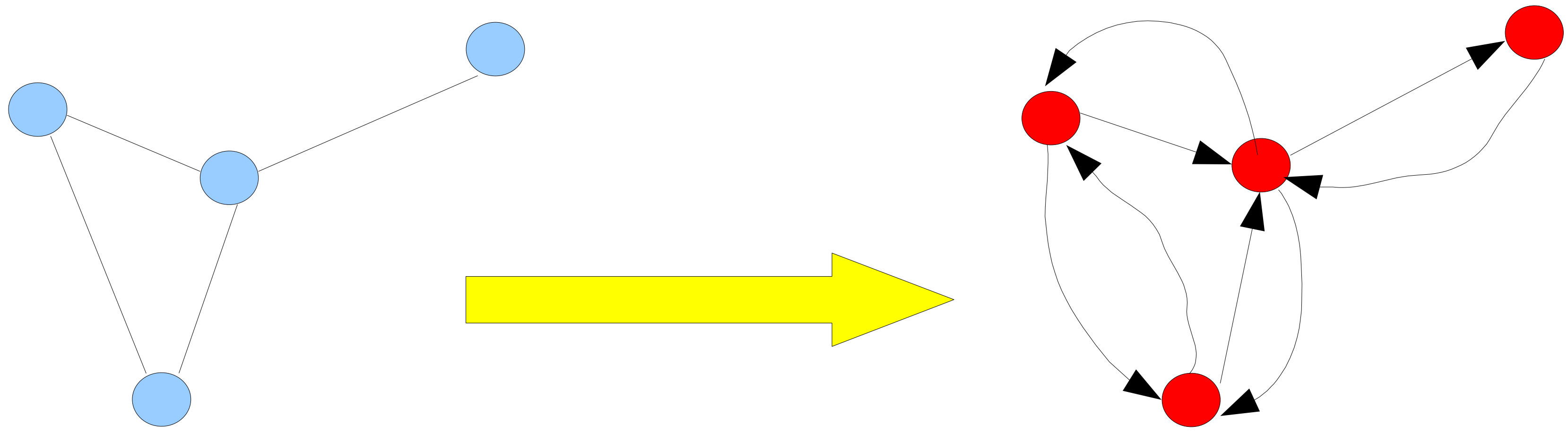
In much he same way as with undirected graphs, we define a **directed trail**, **directed walk**, **directed path** and **directed circle**, which is also called a **cycle**. The only difference is that edges in the corresponding respective defining sequences are replaced by arcs going from previous to subsequent nodes.



Red arcs in the above figure represent a directed path while the blue ones show a cycle.

Directing a graph symmetrically

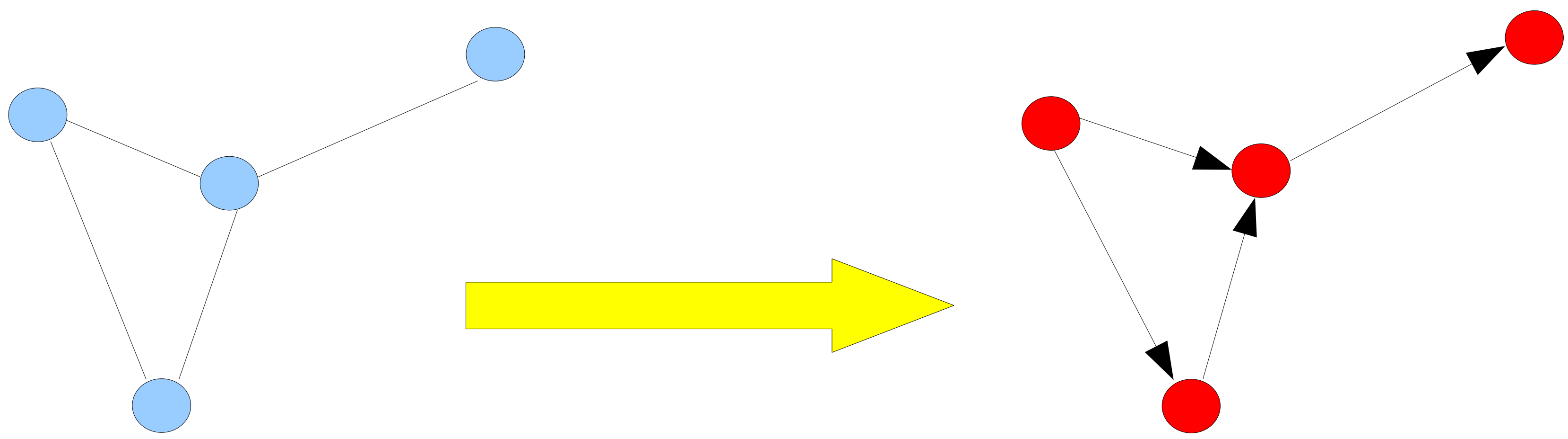
Given a simple graph $G=(N,E)$, a directed graph $G'=(U,E',\epsilon)$ may be defined such that, for every edge $\{u,v\}\in E$ there are in E' exactly two arcs a, a' such that $\epsilon(a)=(u,v)\wedge\epsilon(a')=(v,u)$. Moreover, E' contains no other arcs. We say that such a graph has been created by *symmetrically directing* G. In other words, an edge in a simple graph between u and v is replaced by two arcs between u and v in the new graph.



Directing a graph

Given simple graph $G=(N,E)$ a directed graph $G'=(N,E',\epsilon)$ may be defined such that tak, for any edge $\{u,v\}\in E$ there is in E' a unique arc a such that $\epsilon(a)=(u,v)$ or $\epsilon(a)=(v,u)$ with E' containing no other arcs. We say that such a graph has been created by directing graph G.

In other words, an edge between *u* and *v* is replaced by an arc going from *u* to v or from *v* to *u*. It is clear that, as opposed to a graph create by directing a graph symmetrically, which is unique, a simple graph may be directed in a number of ways. Moreover a directed graph created by directing a simple graph does not contain cycles of length two.

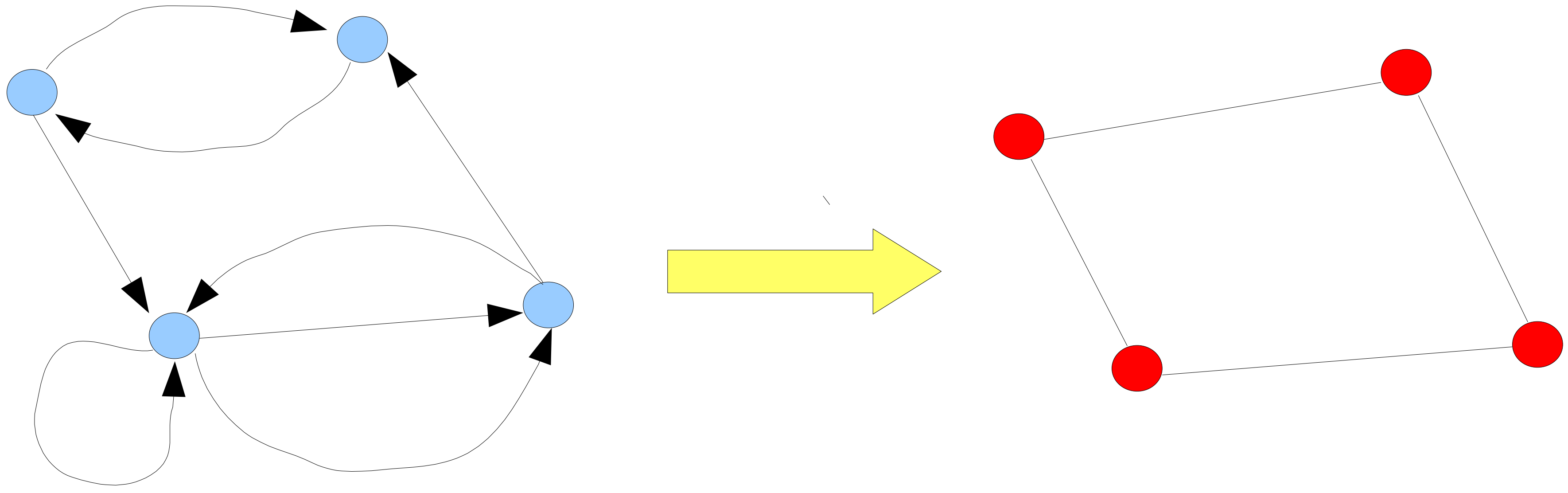


Symmetrizing a digraph

Given a directed graph $G=(N,A,\epsilon)$, there exists a unique simple graph called the *symmetrization of G*. Put

$$A'=\{\{u,v\}|u,v\in N,u\neq v,\exists a\in A:(\epsilon(a)=(u,v)\vee\epsilon(a)=(v,u))\}$$

In other words, arrows, multiple arcs and loops in the original graph are "disregarded".



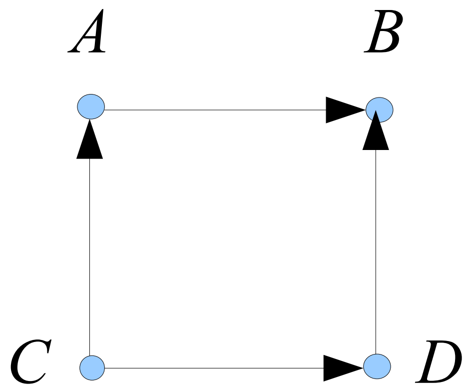
Connectivity

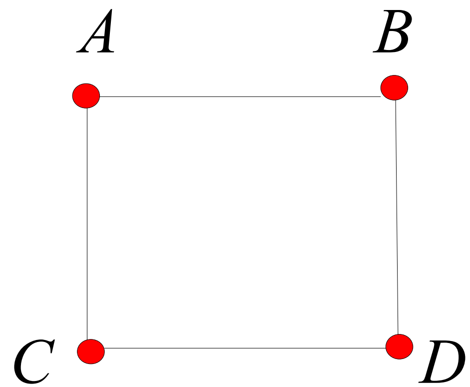
We say that a directed graph $G=(N, A, \epsilon)$ is connected if its symmetrization $G'=(U, H')$ is a connected graph.

Strong connectivity

We say that a directed graph $G=(N, A, \epsilon)$ is strongly connected if, for any two nodes $u, v \in N$, there is a directed path from *u* to *v*.

Clearly, any strongly connected graph is also connected but the opposite may not be true.



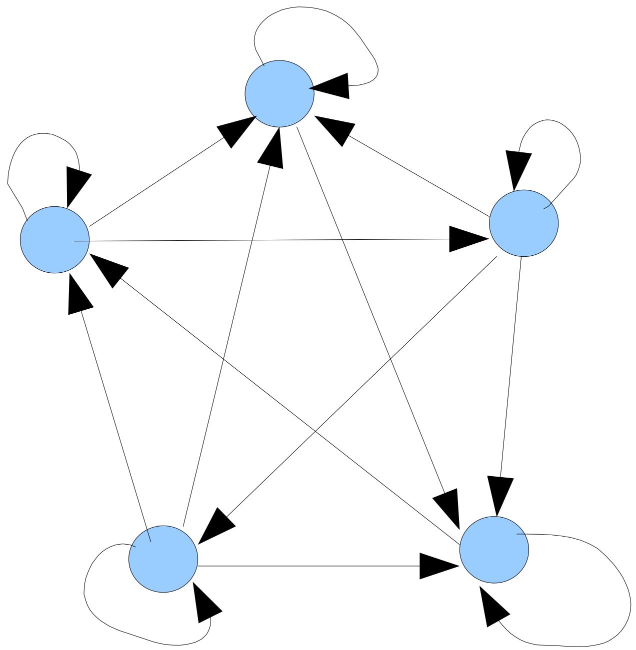


The graph is not strongly connected as there is no directed path starting at *B*.

Tournament

A directed graph $G=(N,A,\epsilon)$ is called a tournament if, for any set of nodes $\{u,v\},u,v\in N,u\neq v$, there exists a single arc $a\in A$ such that $\epsilon(a)=(u,v)\vee\epsilon(a)=(v,u)$

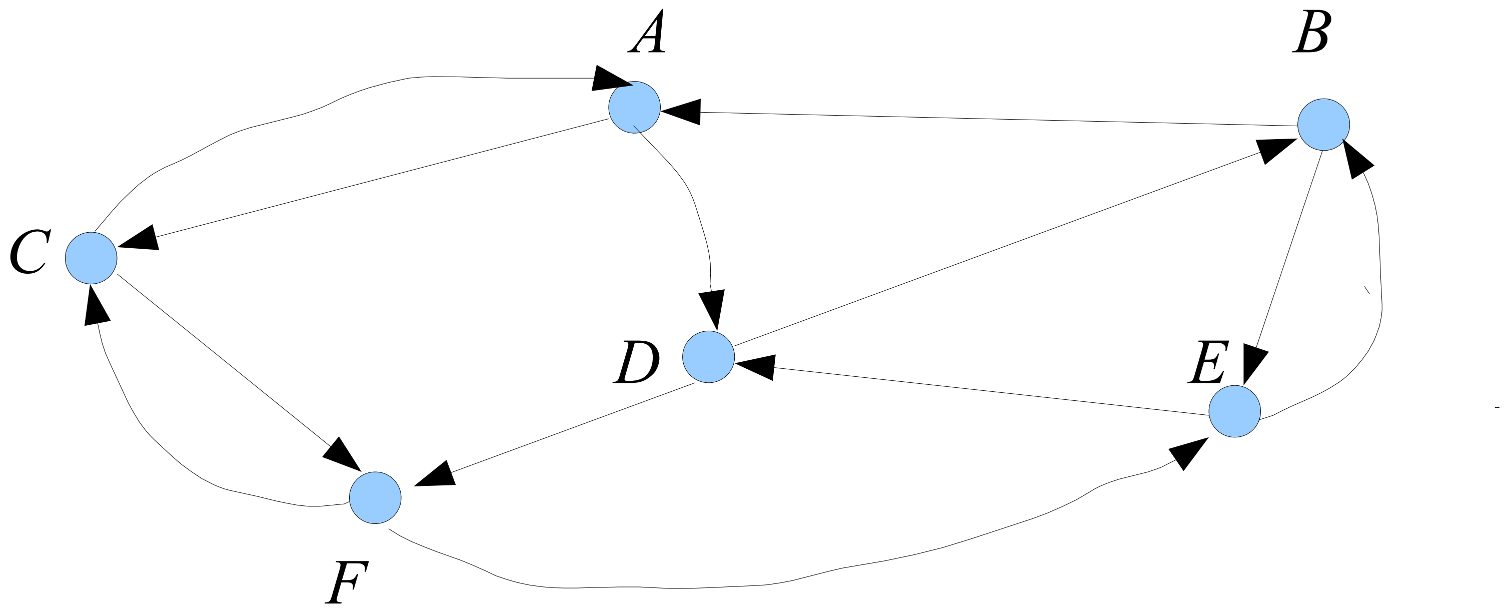
Thus, in a tournament, for any pair of nodes, there is exactly one arc going from one node of the pair to the other.



<u>**Eulerian graph**</u>

A digraph $G=(N, A, \epsilon)$ is called *Eulerian* if there is in G a closed directed walk containing all of its arcs.

Since no identical arcs may appear in a walk, a digraph is Eulerian if and only if all of its arcs can be drawn exactly once without lifting the pen.



For example, in the above graph, we have a walk *C*, *A*, *D*, *F*, *C*, *F*, *E*, *B*, *E*, *D*, *B*, *A*, *C*

Theorem

A connected digraph $G=(N,A,\epsilon)$ is Eulerian if and only if $\deg_+(u)=\deg_-(u)$ for every node $u\in U$.

Proof:

$(\Rightarrow)$ If a digraph is Eulerian, every arc *a* is contained in a single closed walk and so is every node *u*. The assertion of the theorem is a consequence of the fact that, if a node is entered in a closed walk, it must also be left.

$(\Leftarrow)$ If, on the other hand, $\deg_+(u)=\deg_-(u)$ for every $u\in N$, then clearly $\deg_+(u)=\deg_-(u)>0$ since *G* is connected. Let u_0 be any node of *G*. We will prove that there is in *G* a closed walk containing this node. Denote by $t=(u_{-q},h_{-q},u_{-q+1},\ldots,u_{-1},h_{-1},u_{0,}h_{1,}u_{1,}\ldots,u_{p-1},h_p,u_p)$ a walk in *G* of maximal length for which $u_0\neq u_i,-q\leq i\leq-1$ and $u_0\neq u_j,1\leq i\leq p$.

(Clearly, we can assume $q, p > 0$ since $\deg_+(u_0)=\deg_-(u_0)>0$.) Considering that t is maximal and under the assumption, only the following cases are possible:

1) an arc a exists such that $\epsilon(a)=(u_p, u_0)$ and an arc a' such that $\epsilon(a')=(u_{0,} u_{-q})$

2) an arc a'' exists such that $\epsilon(a'')=(u_p, u_{-q})$.

In the first case, for example, $t'=(u_{0,} a_{1,} u_{1,} \ldots, u_{p-1}, a_p, u_p, a, u_0)$ is a closed walk containing u_0 while in the second case, $t=(u_{-q}, a_{-q}, u_{-q+1}, \ldots, u_{-1}, a_{-1}, u_{0,} a_{1,} u_{1,} \ldots, u_{p-1}, a_p, u_p, h'', u_{-q})$ is such a closed walk.

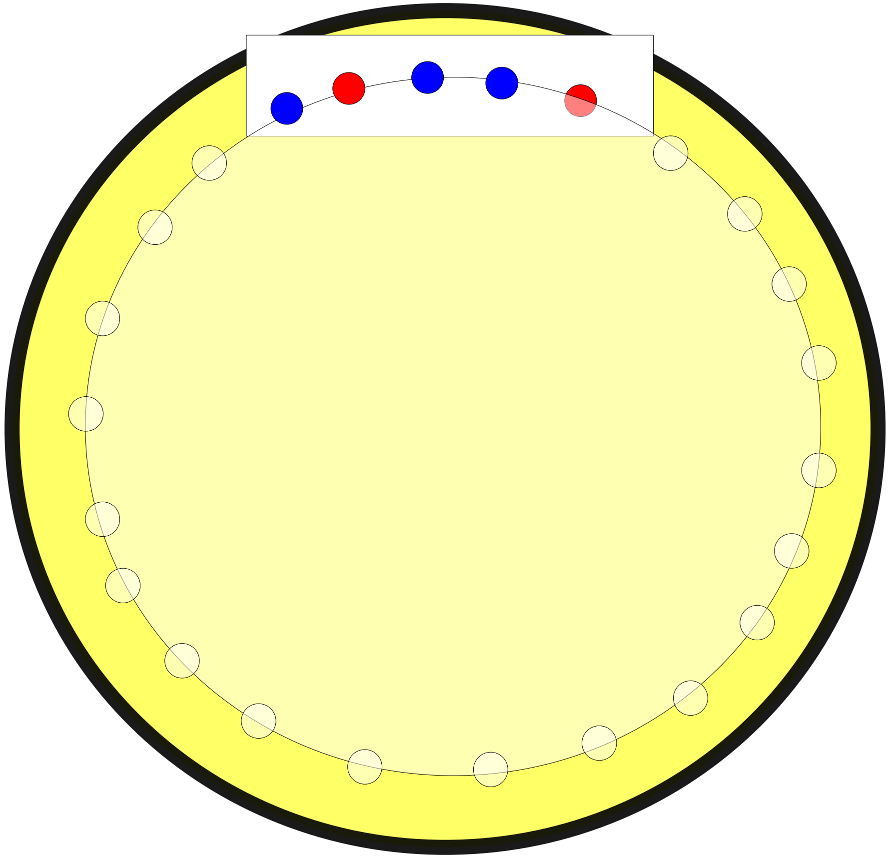
Let now $s=(v_1, g_1, v_2, \ldots, v_{k-1}, g_k, v_k=v_1)$ be a closed walk with a maximum of arcs. Suppose there is an arc not contained in s, then, under the assumption and as G is connected, there must be in s a node v_i which is entered by an arc not contained in *s* and left by an arc not contained in *s*. This means that, in a graph $G'=(N, A', \epsilon)$ with $A'=A-\{g_{1,} g_{2,} \ldots, g_{k,}\}$, we also have $\deg_{+}(u)=\deg_{-}(u)$ for every $u \in N$, $deg_{+}(v_i)=deg_{-}(v_i)>0$ and we can again prove that there is in *G* ' a closed walk $s_i=(v_i, g'_1, w_1, \ldots, w_{s-1}, g'_s, w_s=v_i)$.

Clearly replacing in the walk *s* node u_i by the closed walk s_i, we obtain a closed walk in *G* having more arcs than *s*, which is in contradiction to *s* being maximal. Therefore, the closed walk $s=(v_1, g_1, v_2, \ldots, v_{k-1}, g_k, v_k=v_0)$ contains all the arcs in *A* and *G* is an Eulerian graph.

Problem

Red and blue points are placed at equal distances along the perimeter of a disc in such a way that you can always tell the position of the rotating disc seeing only k consecutive points through a rectangular peephole. Given a k we should design a disc of a maximum perimeter $u(k)$.

$k = 5$

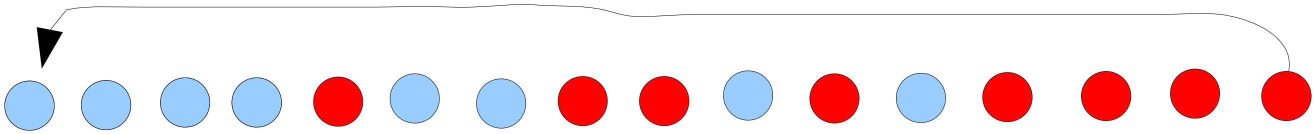


Solution

As there are 2^k different sequences of red and blue points of length k, we have $u(k) \leq 2^k$. We can construct a cyclic ordering of length 2^k having the required properties as follows: Define a digraph $G = (N, A, \epsilon)$ where N is the set of all the sequences of red and blue points of length $k-1$ and A is the set of all the sequences of red and blue points of length k with $\epsilon(a) = ((p_{1,} p_{2,} \ldots, p_{k-1}), (p_{2,} p_{3,} \ldots, p_k))$ for $a = (p_{1,} p_{2,} \ldots, p_k) \in A$. G is an Eulerian graph since $\deg_+(x) = \deg_-(x) = 2$ for every node x of G. It can also be verified that G is connected. Also the number of arcs in G is 2^k. Put $2^k = K$. For every closed walk $(u_{0,} a_{1,} u_{1,} a_{2,} \ldots, u_{K-1}, a_K, u_K)$, we can define a cyclic ordering $(a_{1,}^1 a_{1,}^2 \ldots, a_1^K)$ denoting $a_i = (a_{1,}^i a_{2,}^i \ldots, a_k^i)$ (that is, taking the initial term of each sequence a_i). Due to the choice of nodes and arcs of G, every sequence of length

k occurs in the cyclic ordering $\left(a^1_{1,} a^2_{1,} \ldots, a^K_1\right)$.

For $k = 4$, for example, we have the following cyclic ordering

,

which can be obtained from a suitable closed walk in the digraph bellow.