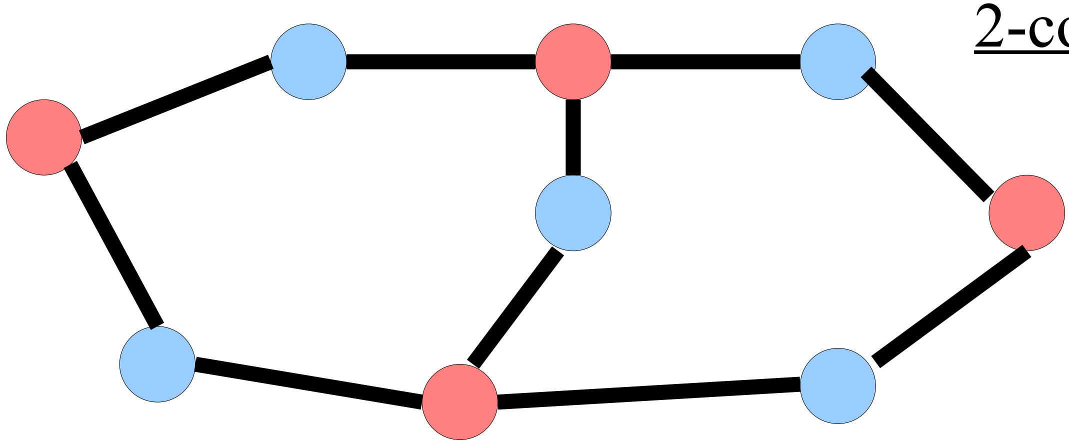
**Node Colouring** (simple graphs without loops)

A graph is said to be ***coloured*** if each vertex is assigned a colour in such a way that any two adjacent nodes are assigned different colours.

If, in a graph, such an assignment is possible using at most $k$ colours, we call the graph ***k-colourable.***

The smallest value of $k$ for which a graph $G$ is $k$-colourable is the chromatic number of $G$, formally denoted by $\chi(G)$ .

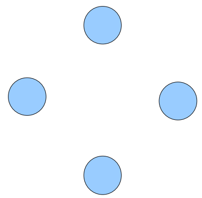
2-colourable graph



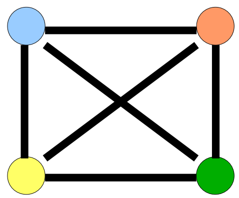
Let $K_n$ denote a complete graph with $n$ nodes, $D_n$ graph with $n$ nodes and no edges, and $K_{m,n}$ a bipartite graph with $m+n$ nodes.

<u>The following assertions can be proved easily:</u>

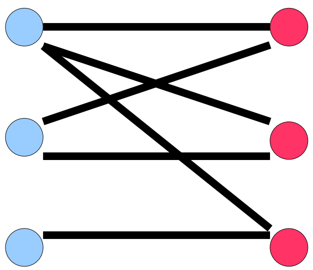
(a) $\chi(G)=1 \Leftrightarrow G=D_n$



*(b)* $\chi(K_n)=n$

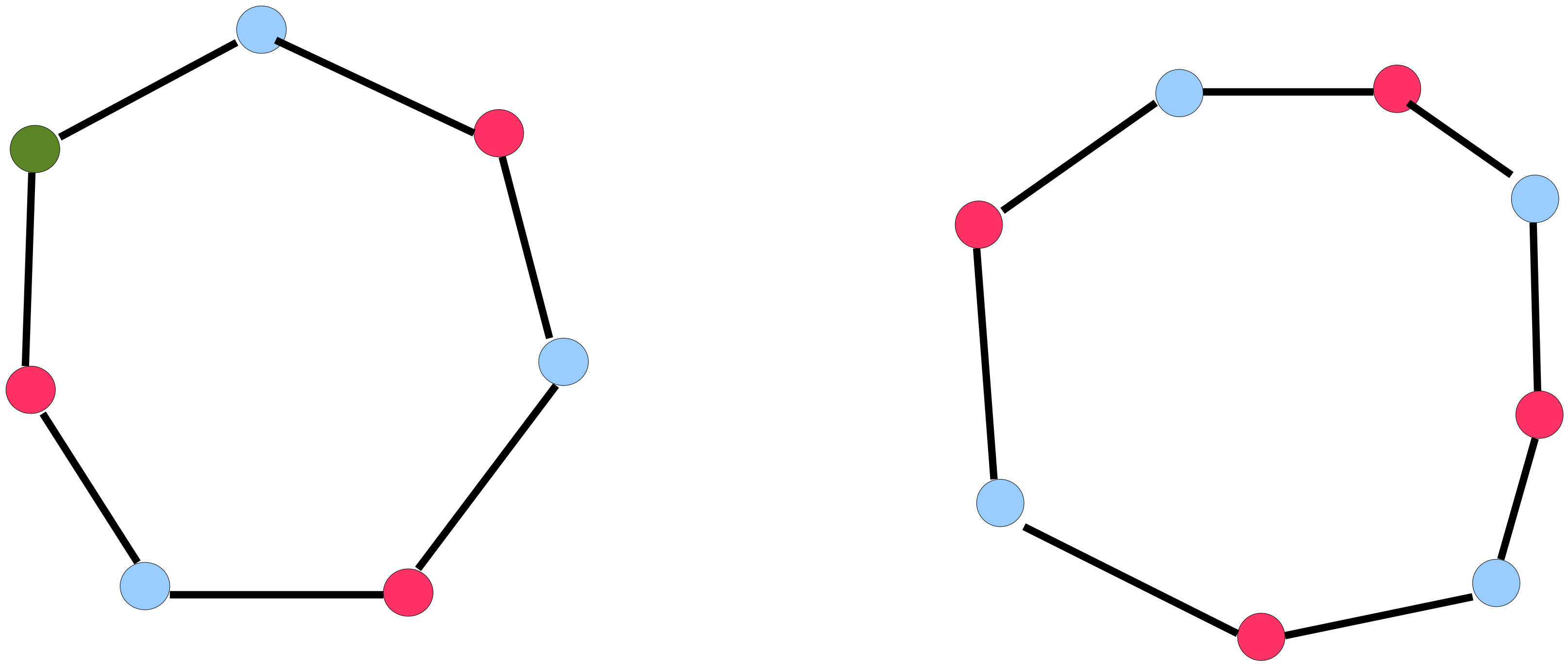


*(c)* $\chi(K_{m,n})=2$



The following assertion is also easy to see:

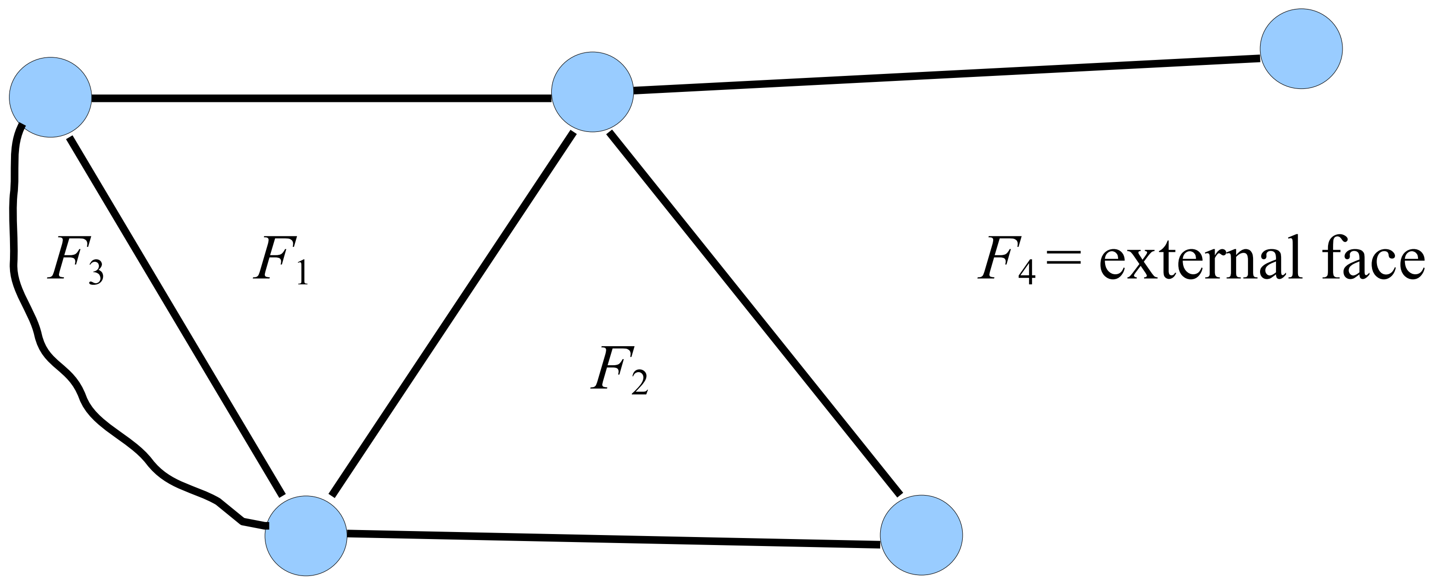
*A circle is 2-colourable iff it has an even number of nodes*



Consequently, a graph containing no odd circle is 2-colourable and one containing an odd cycle is not 2-colourable. A tree, which is known to contain no circle at all, is thus 2-colourable.

## Graph planarity

A graph G (not necessarily simple) is called ***planar***, if it **can be drawn** in a plane so that any two of its edges may only intersect at a node. A planar graph **drawn** in this way is called a ***plane graph***. The two-dimensional region bounded by edges in a plane graph is called a face and the nodes and edges around a face are its boundaries.



**THEOREM 1** (*Euler 1750*)

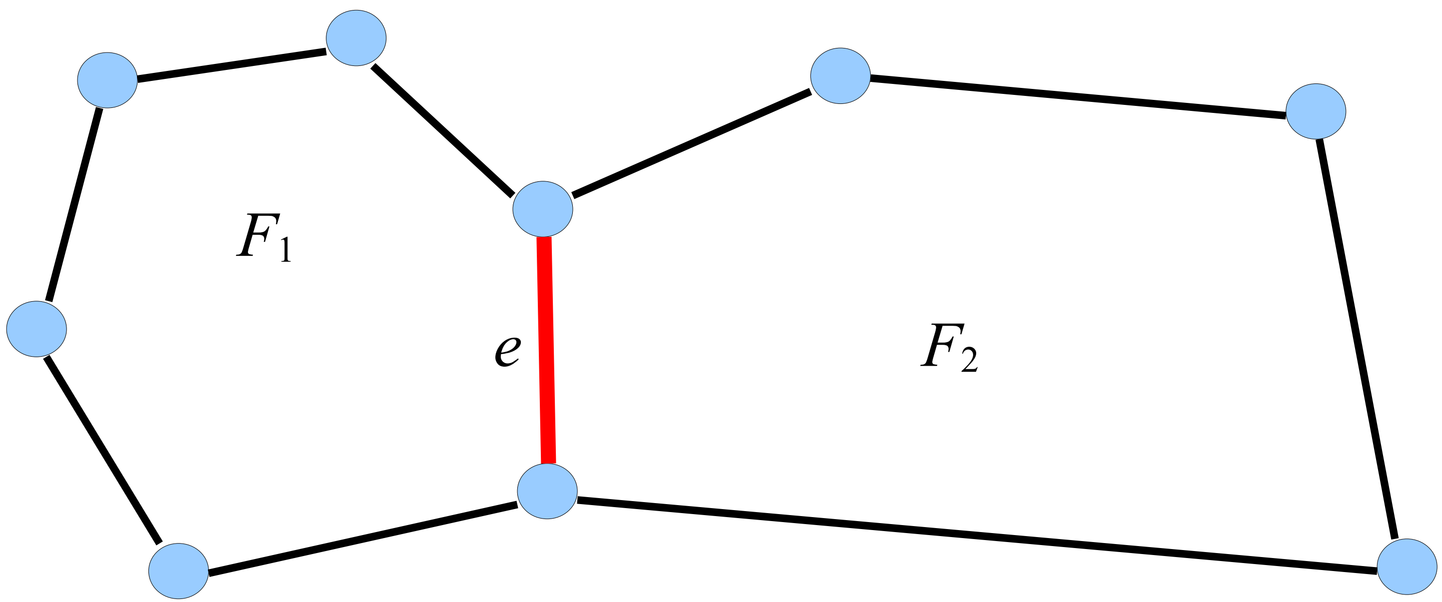
If a connected plane graph has $n$ nodes and $m$ edges and forms $p$ faces, then

$$n - m + p = 2$$

Proof:

Let $G$ be a plane, connected graph. We will proceed by induction on the number of edges. If $m = 0$, then $n = 1$ since $G$ is connected and $p = 1$ so that the equation holds. Let the equation hold for any $m = k - 1$. Let $G$ be a graph with $n$ nodes and $k$ edges. If $G$ is a tree, the equation clearly holds (every tree with $n$ nodes has exactly $n - 1$ edges and encloses no face). If it is not a tree, it contains a circle $C$. Let $e$ be an edge of $C$ and let us delete it forming graph $G'$. $G'$ is still be connected, has $n$ nodes and $k - 1$ edges.

Thus , it has $2-n+k-1=1-n+k$ faces. However, removing an edge from a circle causes the face enclosed by it to merge with the other face having $e$ as its boundary. Thus the number of faces of $G$ is by one greater than that of $G'$ and $G$ has $2-n+k$ faces.

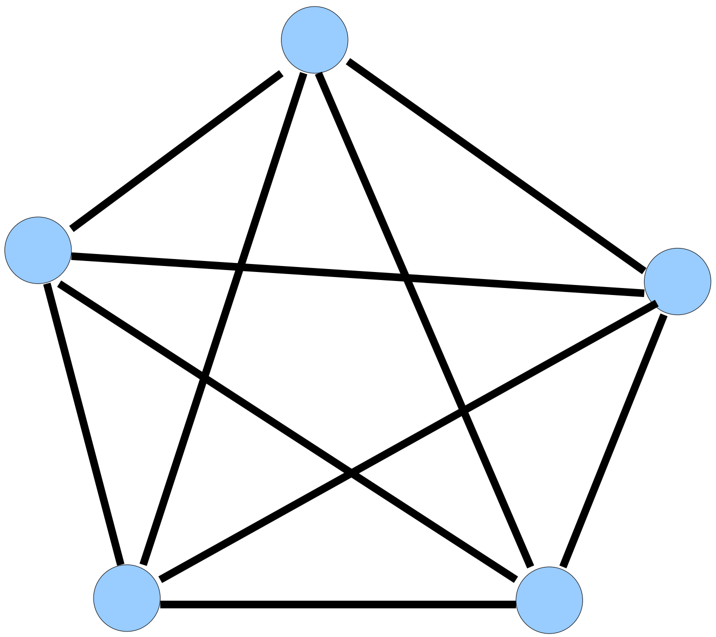


## <u>THEOREM 2</u>

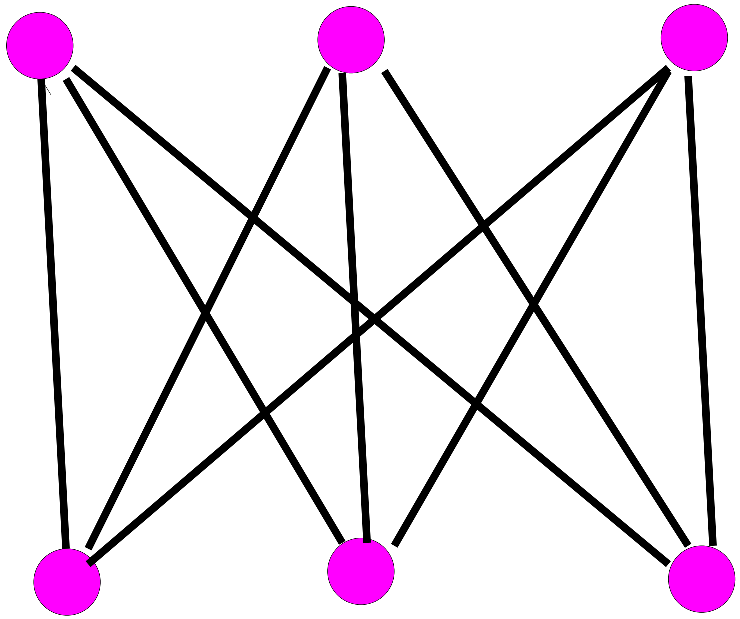
*Let G be a simple connected planar graph with n nodes and m edges. Then* $m \leq 3n - 6$ .

<u>Proof</u>: If $n = 3$ , then the assertion is clear. Let us draw $G$ so that it has faces $F_{1,} F_{2,} \ldots F_p$ . Let $r_i$ be the number of edges that define face $F_i$. Since $G$ is simple, $r_i$ is at least three. This means $3p \leq (r_1 + r_2 + \cdots + r_p)$ . Now, in counting the total number of edges in the boundaries, each edge is counted at most twice. Thus the right side of the inequality is at most $2m$. By Theorem 1, we have $3(2 - n + m) = 3p \leq 2m$ , which proves the theorem.

We will use the above theorem to prove the non-planarity of the following graphs:



$K_5$



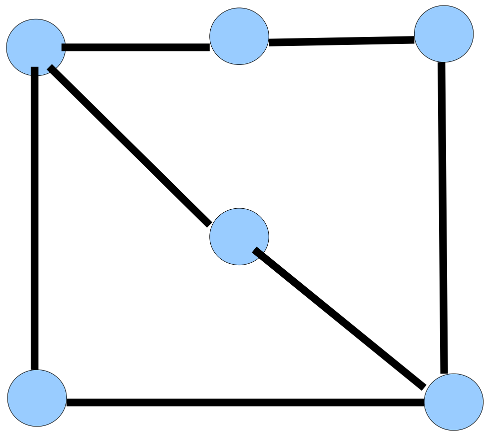
$K_{3,3}$

The non-planarity of $K_5$ follows directly from Theorem 2 since it has 10 edges but a planar graph with such a number of edges cannot have more than 9 edges.

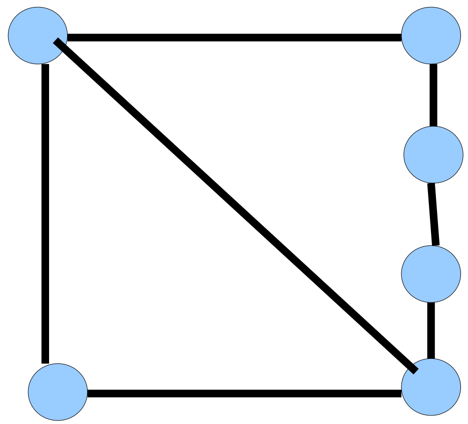
The non-planarity of $K_{3,3}$ can be proved by contradiction. Suppose that it is planar. Then it can be established that it has exactly 5 faces using Theorem 1 with $n = 6$ and $m = 9$. Recall that a bipartite graph has no odd circles. Therefore there must be at least 20 edges to define the boundaries of those 5 faces. Each edge is counted at most twice so that $K_{3,3}$ must have at least 10 edges. This is a contradiction since it only has 9 edges.

*Thus, any graph that has $K_5$ or $K_{3,3}$ as a subgraph is non-planar.*

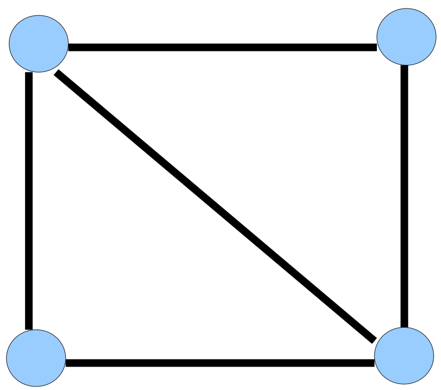
Two graphs $G_1$ and $G_2$ are said to be ***homeomorphic*** (or identical up to nodes of degree 2) if both $G_1$ and $G_2$ can be obtained from a graph $G_3$ by introducing new nodes of degree 2 on its edges.



$G_1$



$G_2$



$G_3$

**THEOREM 3**

*A graph is planar if and only if it contains no subgraph homeomorphic to $K_5$ or $K_{3,3}$.*

This theorem was proved by Kuratowski in 1930.

For a proof, see, for example

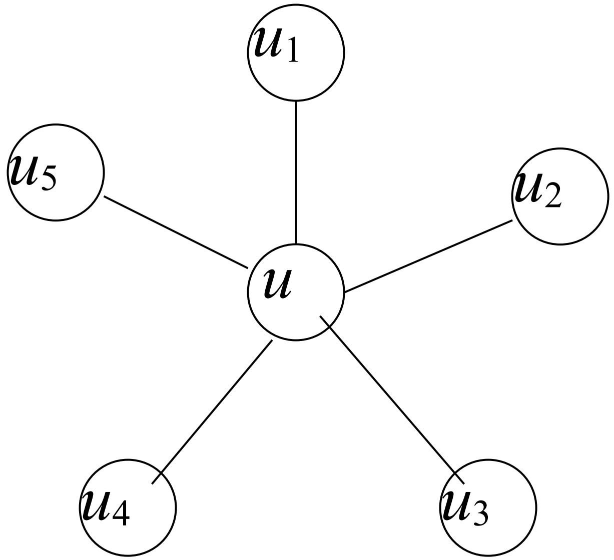
BONDY, J. A., and MURTY, U. S. R. *Graph Theory with Applications*, Elsevier, New York, 1976.

**<u>THEOREM 4:</u>**

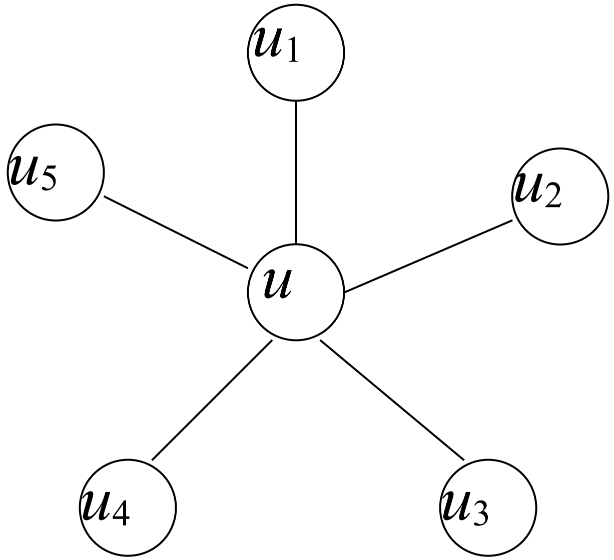
*A planar graph G is 5-colourable.*

**<u>Proof:</u>**

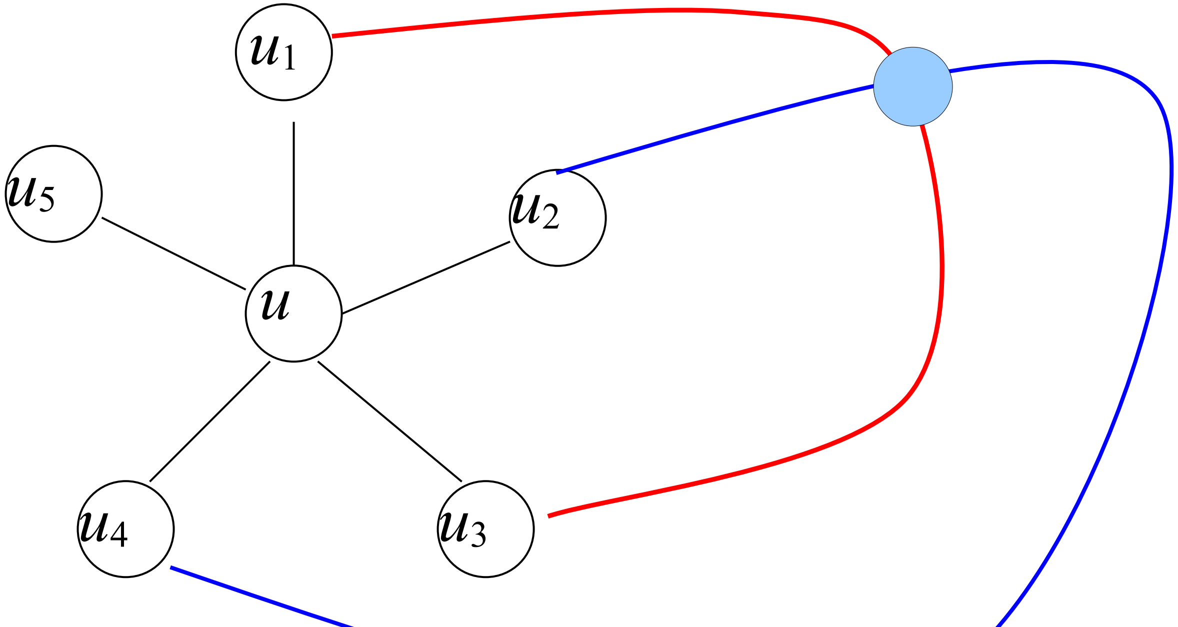
We will proceed by induction on the number of nodes of $G$. Let every graph with less than $n$ nodes be 5 colourable. Consider a graph $G=(N, E)$ with $n$ nodes and construct its colouring using 5 colours. Let $u \in N$ and denote by $G_u$ the graph created from $G$ by removing the node $u$ along with the edges incident on it. As $G_u$ has $n-1$ nodes, it can be coloured using 5 colours. If there are fewer than 5 nodes adjacent to $u$ in $G$, $u$ can be coloured with the colour not assigned to any of its adjacent nodes. Suppose that there are at least 5 adjacent nodes $u_{1,} u_{2,} u_{3,} u_{4,} u_5$ to $u$ in $G$.




Let node $u_i$ be assigned colour $c_i$. If the colours $c_{1,} c_{2,} c_{3,} c_{4,} c_5$ are not distinct, node $u$ can be assigned the colour that is not included in $c_{1,} c_{2,} c_{3,} c_{4,} c_5$ which provides the desired colouring of $G$. Suppose now that $c_i \neq c_j, 1 \leq i < j \leq 5$ and let the node $u$ be assigned, say, colour $c_5$.




Denote by $H_{13}$ the subgraph of $G$ induced by nodes with colours $c_1$ and $c_3$. Similarly, $H_{24}$ will denote the subgraph of $G$ induced by nodes with colours $c_2$ and $c_4$. We will now prove that it is not possible for nodes $u_1$ and $u_3$ to be in the same component of $H_{13}$ if nodes $u_2$ and $u_4$ are in the same component of $H_{24}$.




Suppose that nodes $u_1$ and $u_3$ are in the same component of $H_{13}$, then there is in $G$ a path between them containing only nodes with colours $c_1$ and $c_3$. Since nodes $u_2$ and $u_4$ are in the same component of $H_{24}$, there is in $G$ a path between them containing only the nodes assigned colours $c_2$ and $c_4$. However, this is not possible since $G$ is planar and the colour of the blue node would have to be both in $\{c_1, c_3\}$ and $\{c_2, c_4\}$ .

Let then nodes $u_1$ and $u_3$ be in different components of $H_{13}$ with $u_1$ being, say, in a component $K_1$. We may swap in $K_1$ the assignment of colours to nodes, that is, assign colour $c_3$ to nodes originally assigned colour $c_1$ and vice versa. Clearly, this new assignment with both $u_1$ and $u_3$ being assigned the same colour $c_3$ and node $u$ being assigned colour $c_1$ is a colouring of $G$.

**THEOREM 5**

*A planar graph G is 4-colourable.*

This theorem had been known as the four-colour hypothesis for over 150 years until 1976, when it was proved by Appel and Haken.

Given a geographical map, we can construct a planar graph as follows. Each country is represented by a node. Two nodes are joined by an edge if the two countries have a common border. The minimum number of colours required to colour the map is the chromatic number of the graph thus constructed. Every map gives rise to a planar graph and vice versa.