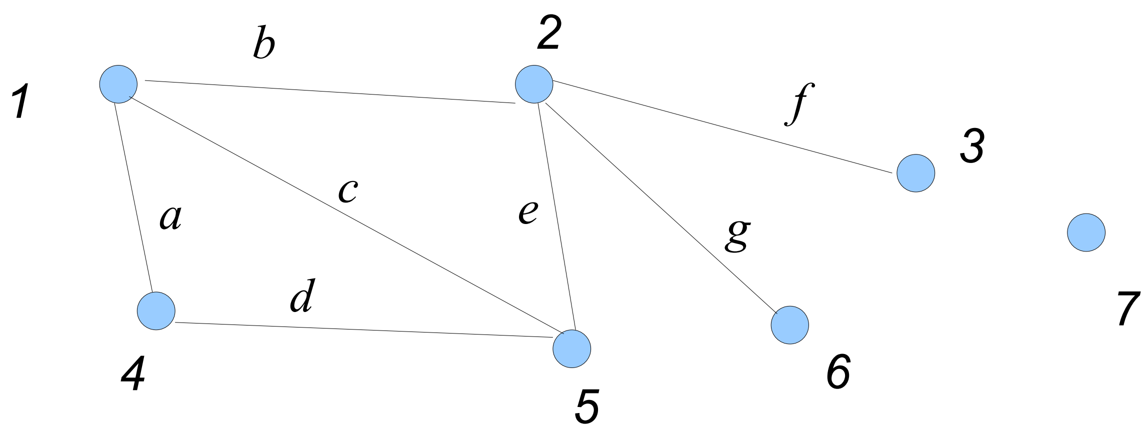
Ordinary (simple undirected) graph

An ordinary graph is a pair $G=(N,E)$ where N is a finite set of nodes (vertices) and $E=\{\{u,v\} : u,v \in N \wedge u \neq v\}$ is a finite set of edges. We say that an edge $e=\{u,v\}$ is incident on nodes u and v or that it connects u and v, etc.

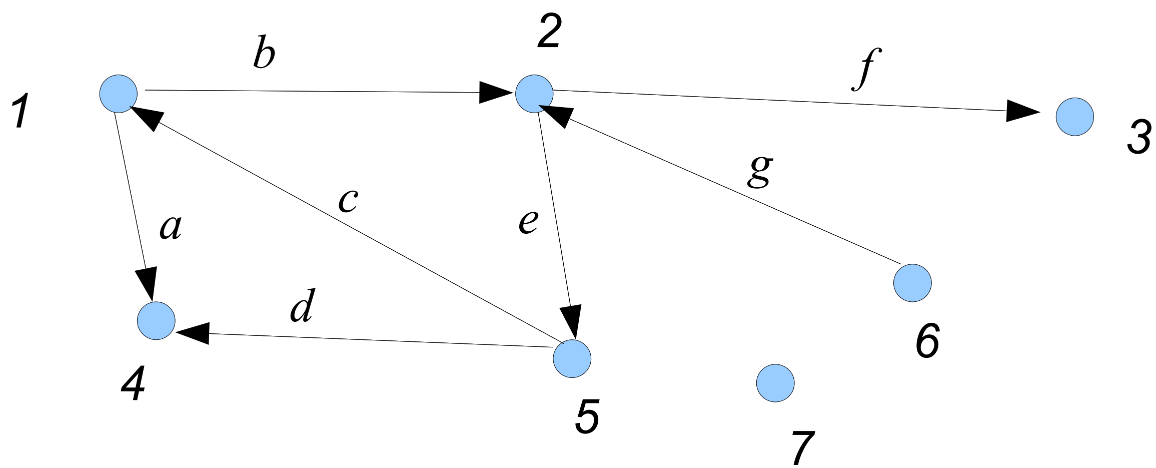


$$a=\{1,4\}$$
$$b=\{1,2\}$$
$$c=\{1,5\}$$
$$d=\{4,5\}$$
$$e=\{2,5\}$$
$$f=\{2,3\}$$
$$g=\{2,6\}$$

$N=\{1,2,3,4,5,6,7\}$ $E=\{a,b,c,d,e,f,g\}$

Simple directed graph

A simple directed simple graph is a pair $G=(N\,,A)$ where N is a finite set of nodes or vertices and $A=\{(u\,,v)\;:\;u\,,v\in N\wedge u\neq v\}$ is a finite set of arcs (ordered pairs).



$a=(1,4)$

$b=(1,2)$

$c=(5,1)$

$d=(5,4)$

$e=(2,5)$

$f=(2,3)$

$N=\{1,2,3,4,5,6,7\}\quad A=\{a\,,b\,,c\,,d\,,e\,,f\,,g\}$

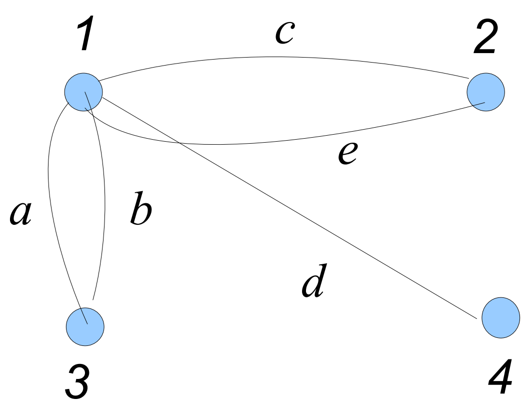
$g=(6,2)$

Graph (multigraph)

A graph is a triple $G=(N, E, \epsilon)$ where N is a finite set of nodes or vertices, E is a finite set of edges and ϵ is a mapping assigning to each edge a pair of different nodes, that is $\epsilon : E \rightarrow \{\{u, v\} : u, v \in N \wedge u \neq v\}$

There may be multiple edges between a single pair of nodes.

$$N=\{1, 2, 3, 4\}$$



$$E=\{a, b, c, d, e\}$$

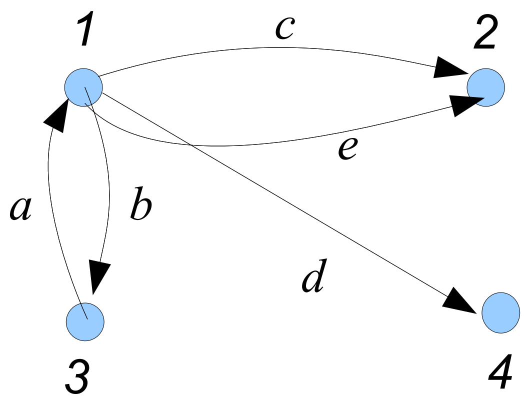
$\epsilon(a)=\{1,3\}$ $\epsilon(b)=\{1,3\}$ $\epsilon(c)=\{1,2\}$

$\epsilon(d)=\{1,4\}$ $\epsilon(e)=\{1,2\}$

Directed graph (directed multigraph)

A directed graph is a triple $G=(N, A, \epsilon)$ where N is a finite set of nodes or vertices, A is a finite set of arcs and ϵ is a mapping assigning to each arc an ordered pair of different nodes: $\epsilon : A \rightarrow \{(u, v) : u, v \in N \wedge u \neq v\}$

There may be multiple edges between a single pair of nodes.



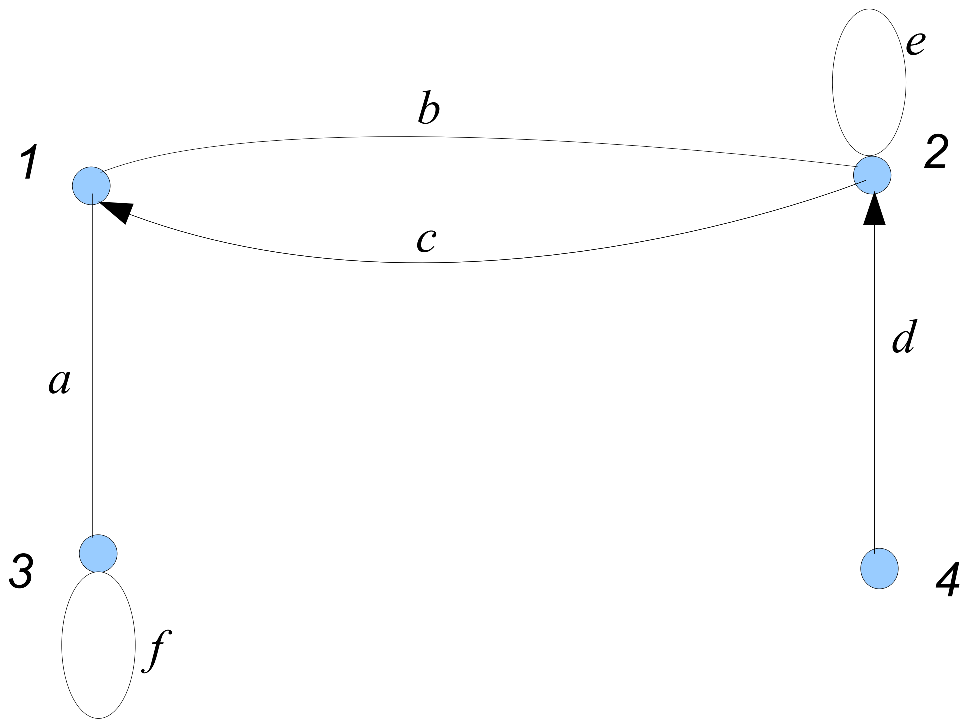
$$N=\{1,2,3,4\}$$

$$E=\{a, b, c, d, e\}$$

$\epsilon(a)=(3,1)$ $\epsilon(b)=(1,3)$ $\epsilon(c)=(1,2)$

$\epsilon(d)=(1,4)$ $\epsilon(e)=(1,2)$

There are also other graph types defined in much the same way. For example, a graph tat has both edges and arcs or a graph with loops, which are edges or arcs beginning and ending in the same node.



$N = \{1, 2, 3, 4\}$

$E = \{a, b, e, f\}$

$A = \{c, d\}$

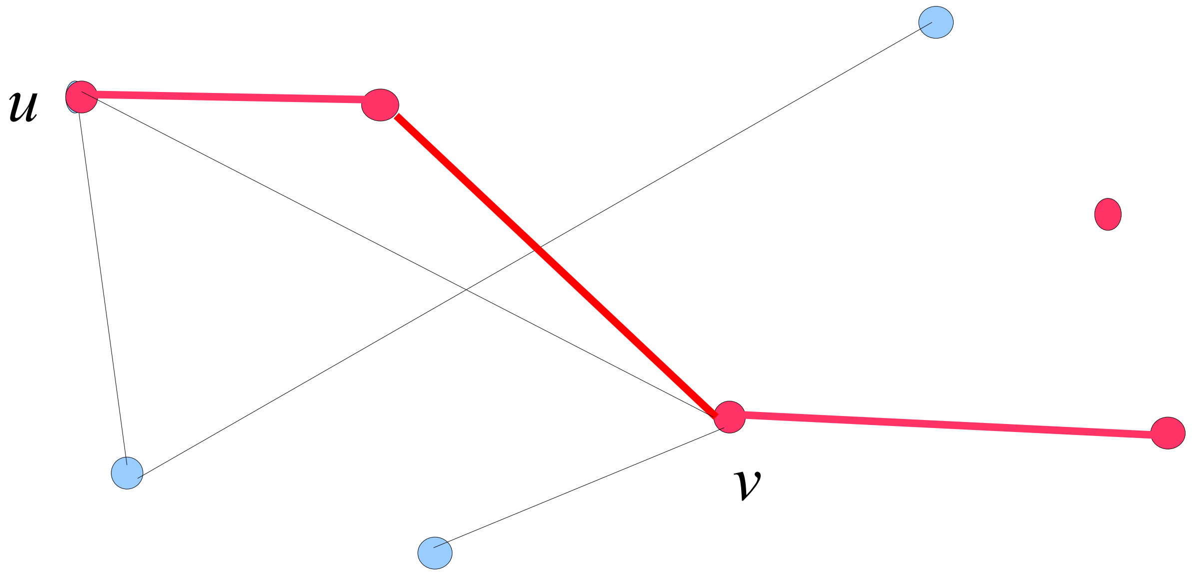
$\epsilon(a) = \{1,3\}$ $\epsilon(b) = \{1,2\}$ $\epsilon(c) = (2,1)$

$\epsilon(d) = (4,2)$ $\epsilon(e) = \{2\}$ $\epsilon(f) = \{3\}$

Subgraph, induced subgraph

If $G=(N,E)$ and $G'=(N',E')$ are ordinary graphs, we say that G' is a subgraph of G or that G contains G', if $N' \subseteq N \wedge E' \subseteq E$

If, moreover, $(u,v \in N' \wedge \{u,v\} \in E) \Rightarrow \{u,v\} \in E'$, G' is called a subgraph of G induced by its nodes.



The red parts of the above graph form its subgraph, however, it is not induced by its nodes as it does not contain edge $\{u,v\}$

Subgraphs and subgraphs induced by its nodes for graphs, simple directed graphs and directed graphs are defined in much the same way.

Trail (walk)

In a graph $G=(N, E, \epsilon)$ we define a trail between nodes u, v of length n as a sequence $(u=w_0, e_1, w_1, e_2, \ldots, w_{n-1}, e_n, w_n=v)$

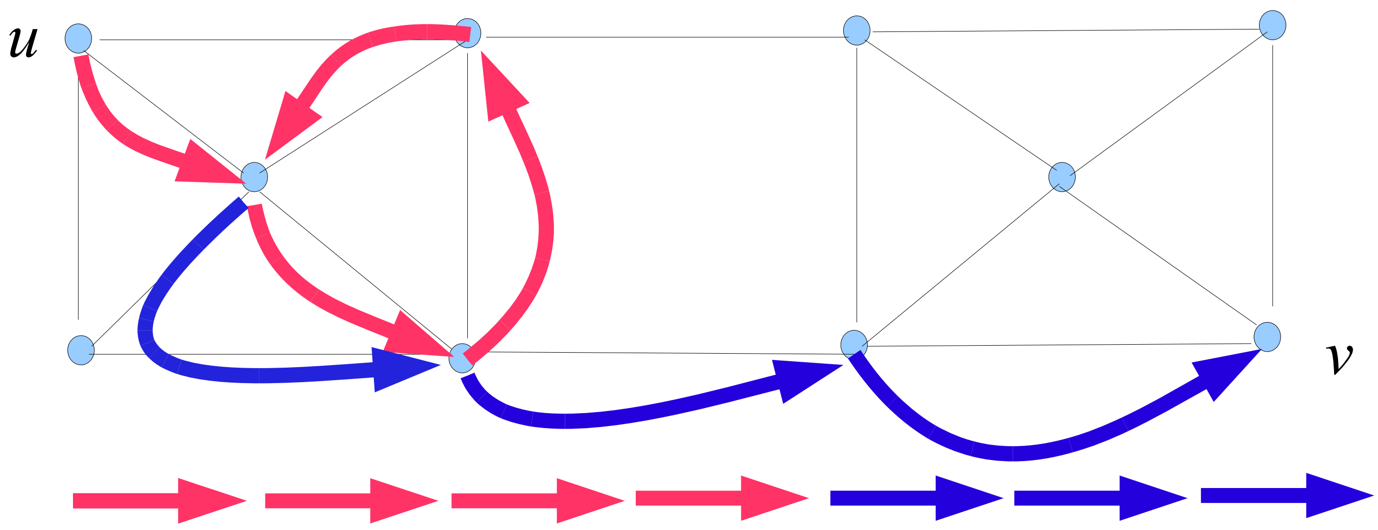
such that

$$w_0, w_1, \ldots, w_n \in N \ , \ e_1, e_2, \ldots, e_n \in E$$

and

$$\epsilon(e_i)=\{w_{i-1}, w_i\}, 1 \le i \le n$$

Thus a trail between nodes u and v of length n is an alternating sequence of nodes and edges beginning with node u, ending with node v and containing n edges with every two neighbouring nodes in the sequence being connected by the intervening edge. In a trail, both nodes and edges may repeat.

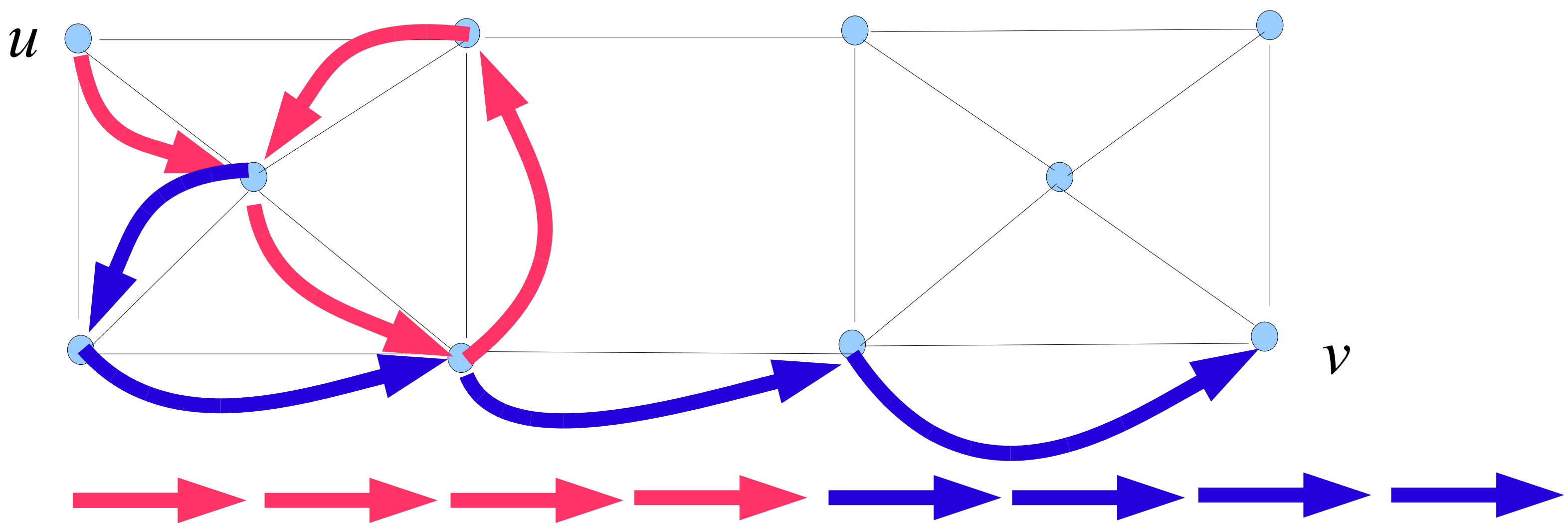


Walk (path)

In a graph $G=(N,E)$ we define a walk between nodes u, v of length n as a trail $(u=w_0, e_1, w_1, e_2, \ldots, w_{n-1}, e_n, w_n=v)$ such that

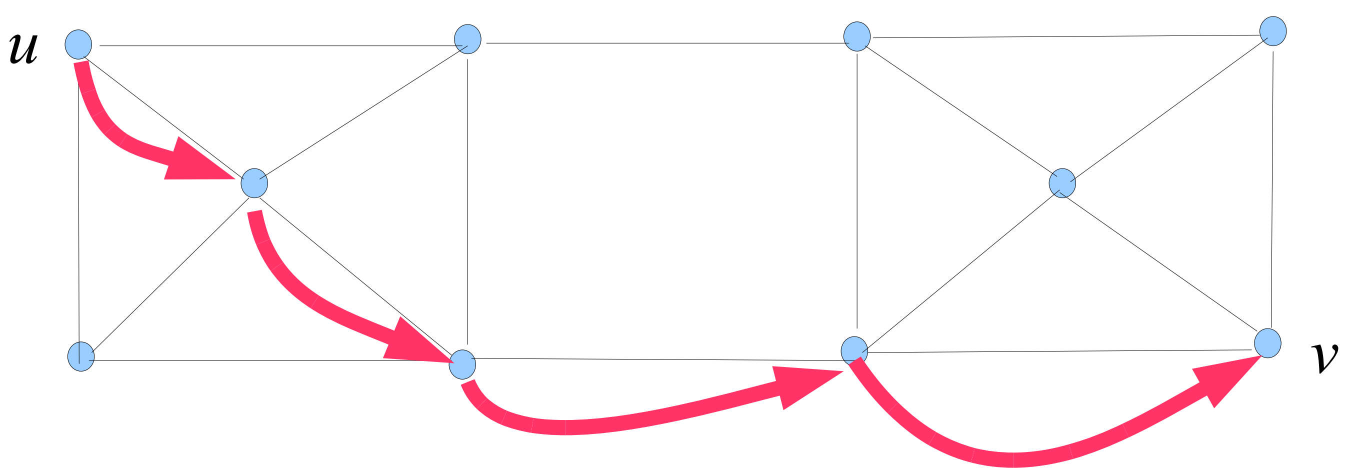
$$i \neq j \Rightarrow e_i \neq e_j, 1 \leq i, j \leq n$$

Thus a walk between nodes u and v of length n is a trail between u, v of length n in which nodes may repeat but all the edges are different.



Path (simple path)

In a graph $G=(N,E)$ we define a path between nodes u, v of length n as a trail $(u=w_0, e_1, w_1, e_2, \ldots, w_{n-1}, e_n, w_n=v)$ between u and v such that $i \neq j \Rightarrow w_i \neq w_j, 0 \leq i, j \leq n$



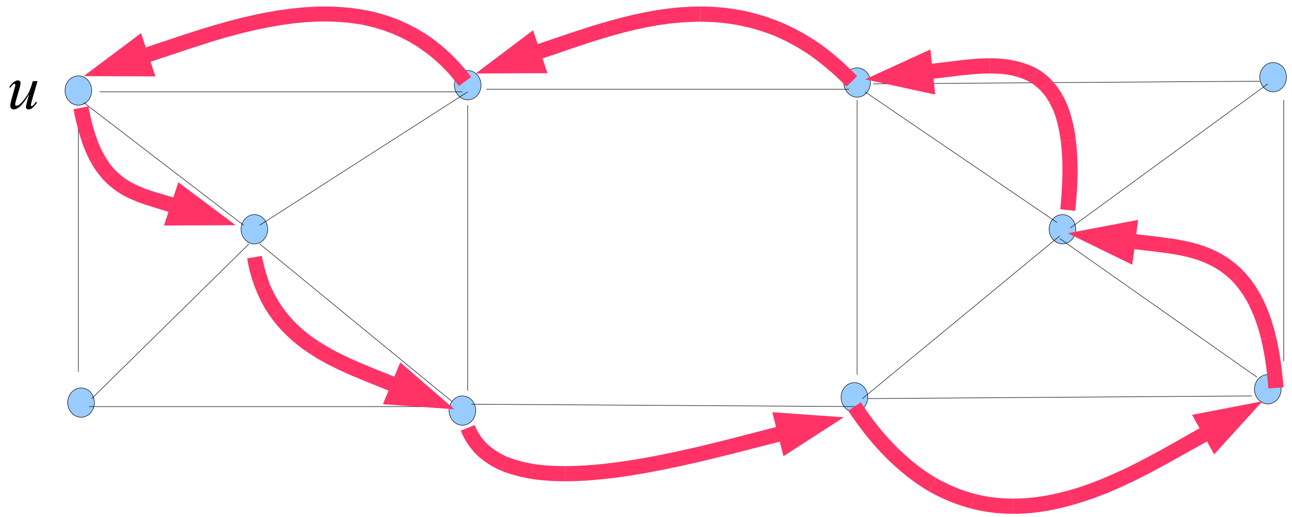
- Thus a path between nodes u and v of length n is a trail between u and v of length n in which all the nodes are different.
- It is not difficult to show that, in a graph G, there is a path between u and v if and only if (iff) there is a trail between u and v

Circle

In a graph $G=(N, E, \epsilon)$ we define a circle of length n as a trail $(w_{0,} e_{1,} w_{1,} e_{2,} \ldots, w_{n-1}, e_n, w_n)$ such that

$$i \neq j \Rightarrow w_i \neq w_j, 0 \leq i, j \leq n-1 \wedge w_0 = w_n$$

Thus a circle in G of length n is a trail in which all the edges are different and nodes are different, too, with the exception of the first and last nodes, which coincide. Clearly, if G is an ordinary graph, we have $n > 2$.



Theorem 1

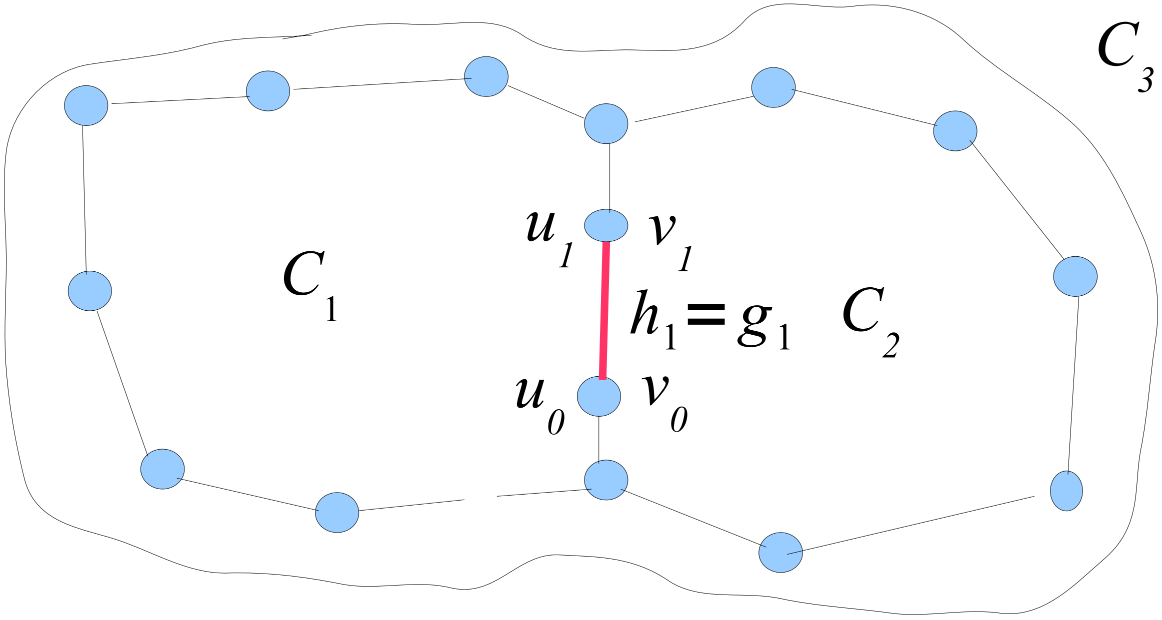
Let an ordinary graph $G=(N, E)$ contain two different circles

$C_1=(u_0, e_{1,} u_1, e_2, u_2, \ldots, u_{p-1}, e_p, u_p=u_0)$ and

$C_2=(v_0, f_{1,} v_1, f_2, f_2, \ldots, v_{q-1}, f_q, v_q=v_0)$

with $u_0=v_0, u_1=v_1, e_1=f_1$. Then this graph also contains a circle C_3

without the edge $e_1=f_1$.



Proof

Let r be the least index such that $u_0 = v_0, u_1 = v_1, \ldots, u_r = v_r$. Clearly $1 \le r \le \min\{p-2, q-2\}$ as C_1 and C_2 are different and G is ordinary. Let s, t be the least indices such that

$$r < s, r < t, u_s = v_t, u_i \neq v_j, r < i < s, r < j < t$$

These indices do exist because $u_{r+1} \neq v_{r+1}$ a $u_p = v_q$. Then the trail

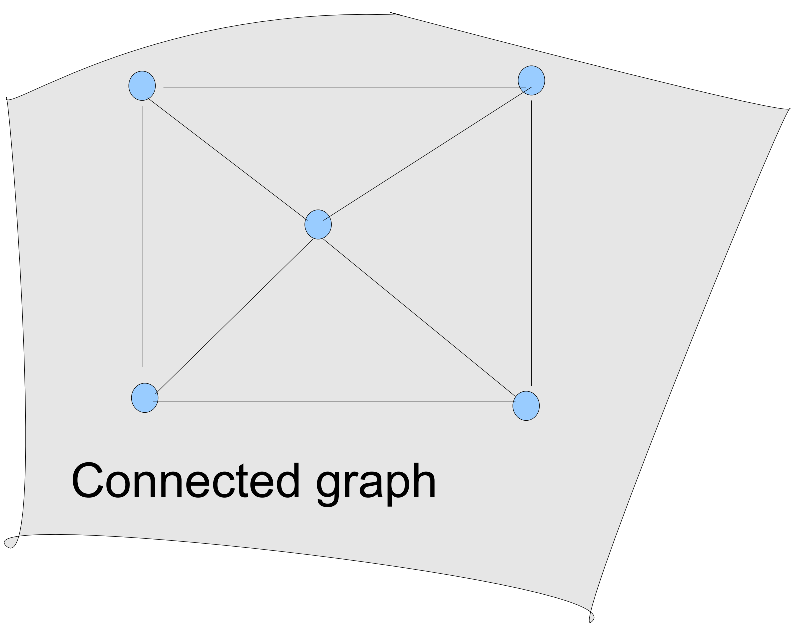
$$u_r, e_{r+1}, u_{r+1}, \ldots, u_{s-1}, e_s, u_s = v_t, f_t, v_{t-1}, \ldots, v_{r+1}, f_{r+1}, v_r$$

is a circle since $u_i \neq v_j$ for $r < i < s, r < j < t$ and $u_r = v_r$. It is easy to see now that this circle does not contain edge $e_1 = f_1$

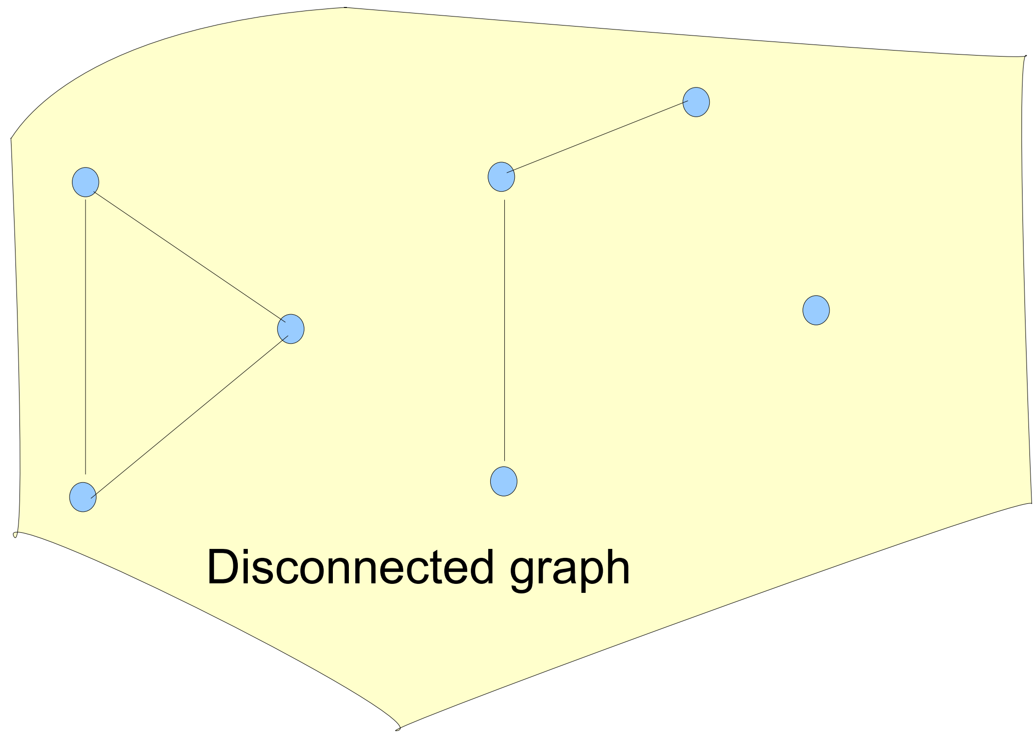
Connected graph

We say that a graph $G=(N,E,\epsilon)$ is connected if, for any pair of its nodes $u,v\in N$ there is a trail(and thus a path, too)

$$(u=w_{0,}e_{1,}w_{1,}e_{2,}\ldots,w_{n-1},e_n,w_n=v)$$



Connected graph

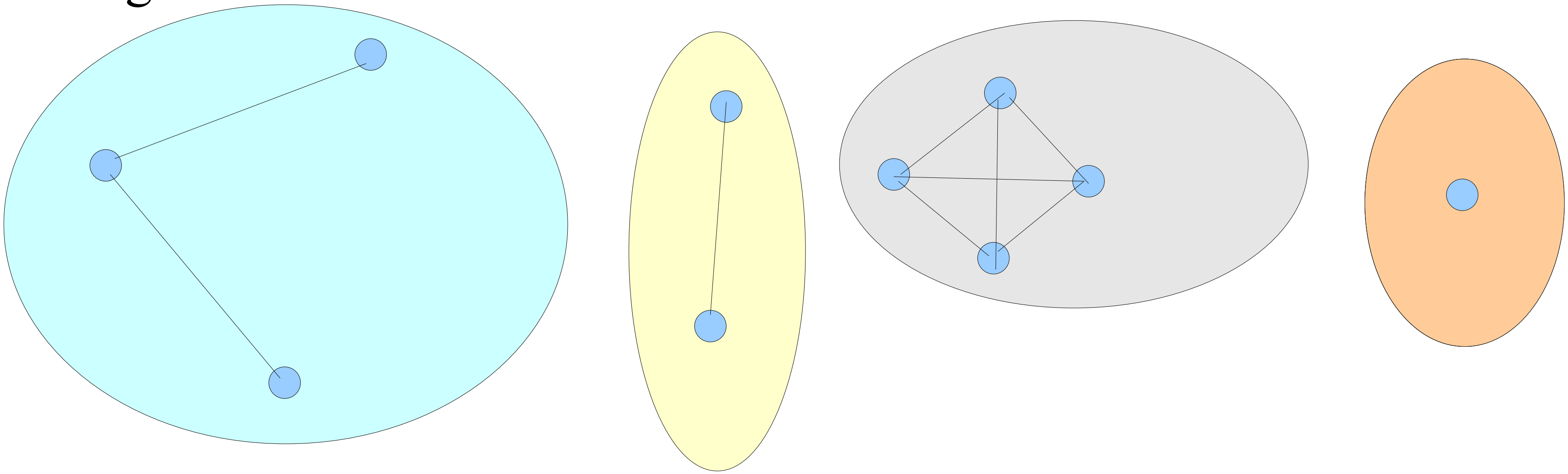


Disconnected graph

Graph and its components

For graphs $G=(N,E,\epsilon)$ and $G'=(N',E',\epsilon)$ we say that G' is a component of G if G' is a connected subgraph of G induced by its nodes and any subgraph $G''=(N'',E'',\epsilon)$ of G such that $N'\subset N''$ and $E'\subseteq E''$ is disconnected

A component of a graph is its subgraph induced by its nodes with the largest number of nodes.

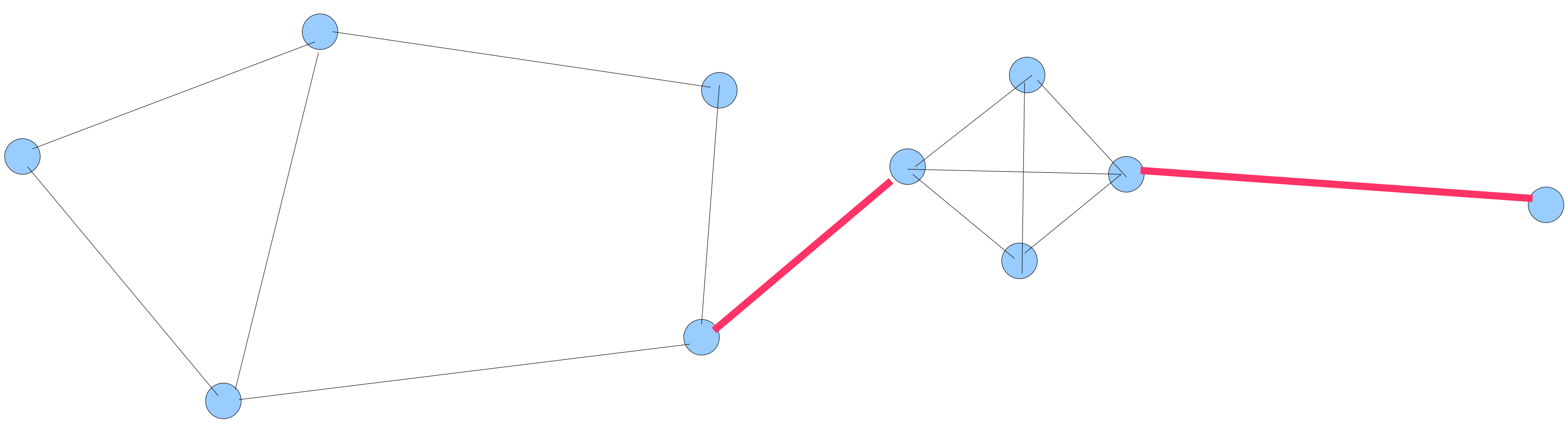


The above graph has four components.

Bridge

We say that an edge e in a graph $G=(N, E, \epsilon)$ is a bridge if the number of components increases after edge e is removed.

If e is an edge in G with $\epsilon(e)=\{u, v\}$ and it is a bridge in G, after its removal, the nodes u and v will lie in different components of G.



The red edges above are bridges.

Node degree

For a graph $G=(N,E,\epsilon)$ and a node $u\in N$ we define the degree of u as

$$\deg(u)=\left|\left\{e\in E \mid \epsilon(e)=\{u,v\}\right\}\right|$$

In other words, the degree of a node is the number of edges incident on it.

It is not difficult to prove that, for a graph $G=(N,E,\epsilon)$ such that $|E|=m$

$$\sum_{u\in N}\deg(u)=2m$$