1\. $y' = x^3 + 3$ 1. ZÁKLADNÍ typ $y' = f(x)$.

$x, C \in \mathbb{R}$ **1**

Ř: $\frac{dy}{dx} = x^3 + 3$

$$\int dy = \int (x^3+3)\,dx$$

$$\underline{y = \frac{x^4}{4} + 3x + C}$$

2\. $y' = e^x - \sin 2x$

Ř: $\frac{dy}{dx} = e^x - \sin 2x$

$$\int dy = \int e^x - \sin 2x \, dx$$

$$\underline{y = e^x + \frac{1}{2}\cos 2x + C}, \quad x \in \mathbb{R}, \ C \in \mathbb{R}$$

3\. $y' = \frac{1}{x-2}$ $\quad x \neq 2$

Ř: $\frac{dy}{dx} = \frac{1}{x-2}$

$$\int dy = \int \frac{1}{x-2}\,dx$$

$$\underline{y = \ln|x-2| + C, \quad x \neq 2}$$

4. $y' = \sqrt{3x}$

Ř: $\frac{dy}{dx} = \sqrt{3x}$

$$\int dy = \sqrt{3}\int \sqrt{x}\,dx$$

$$y = \sqrt{3}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$y = \frac{2\sqrt{3}}{3}\sqrt{x^3} + C\,, \quad x \geq 0,\ C \in \mathbb{R}$$

---

5. $y' = \frac{1}{2x-6}$

Ř: $\frac{dy}{dx} = \frac{1}{2x-6}$

$$\int dy = \frac{1}{2}\int \frac{1}{x-3}\,dx$$

$$y = \frac{1}{2}\ln|x-3| + C$$

$$y = \ln\sqrt{|x-3|} + C\,, \quad x \neq 3,\ C \in \mathbb{R}$$

Cauchyho úloha

6. $y' = 3x^2 - 4x\,, \quad y(0) = 1$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\int dy = \int (3x^2 - 4x)\,dx$$

$$y = x^3 - 2x^2 + C\,, \quad x, C \in \mathbb{R}$$

$y(0) = 1$

$0^3 - 20^3 + C = 1$

$C = 1$

$y = x^3 - 2x^2 + 1$

Nyní určíme hodnotu konstanty C

7. $y' = \frac{-2}{x^3}$, $y(0) = 2$

Ř: $\frac{dy}{dx} = \frac{-2}{x^3}$

$$\int dy = -2\int \frac{1}{x^3}\,dx$$

$$y = \frac{1}{x^2} + C \quad , x \neq 0,\ C \in \mathbb{R}$$

! Hodnotu konstanty C s využitím počátečních podmínek nelze určit (nelze dosadit, dělení nulou)

Závěr: Hledané partikulární řešení neexistuje

Domácí cvičení

D1: $y' = x^4 + 4$ V: $y = \frac{x^5}{5} + 4x + C$

D2: $y' = \frac{1}{x^5}$ V: $y = -\frac{1}{x^4} + C$

D3: $y' = (3x+1)^4$ V: $y = \frac{1}{15}(3x+1)^5 + C$

D4: $y' = \frac{1}{3}\sin\frac{x}{3}$ V: $y = -\cos\frac{x}{3} + C$

D5: $y' = x^6 - 2x$, $y(1) = 0$ V: $y = \frac{x^7}{7} - x^2 + \frac{6}{7}$

D6: $y' = \frac{1}{(x-6)^2}$, $y(7) = 1$ V: $y = \frac{-1}{x-6} + 2$ $x \neq 6$

D7: $y' = \frac{1}{x}$, $y(0) = 5$ V: neexistuje

D8: $y' = \frac{1}{x+3}$, $y(-2) = 4$ V: $y = \ln|x+3| + 4$ $x \neq -3$

2. Separované proměnné

$$p(x) + q(y)\cdot y' = 0$$

$$\Downarrow \quad p(x)\,dx + q(y)\,dy = 0$$

$$\int p(x)\,dx + \int q(y)\,dy = 0$$

1. $y^2 y' = x - 2$

Ř: $y^2 \dfrac{dy}{dx} = x - 2$

$$\int y^2\,dy = \int (x-2)\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 2x + C$$

$$y^3 = \frac{3}{2}x^2 - 6x + 3C$$ opět konstanta $3C = k$

$$y = \sqrt[3]{\frac{3}{2}x^2 - 6x + k}$$

2. $\dfrac{y'}{y} = 4x$

Ř: $\dfrac{1}{y}\dfrac{dy}{dx} = 4x$

$$\int \frac{1}{y}\,dy = \int 4x\,dx$$

$$\ln|y| = 2x^2 + C$$

$$e^{\ln|y|} = e^{2x^2+C}$$

$$|y| = e^{2x^2+C}$$ $k = e^C$

$$y = \pm e^C \cdot e^{2x^2}$$

$$y = k\cdot e^{2x^2}, \quad k \neq 0$$

PŘÍKLADY na nulové řešení

3. $y' = 4xy$

Ř: $\frac{y'}{y} = 4x$ dělení nulou $y \neq 0$
Tím musíme $y = 0$ (singul. řešení = nulovou funkci doplnit)

stejně jako v př. 2

$$y = k \cdot e^{2x^2}, \; k \in \mathbb{R}$$

4. $y' = 3 \cdot \sqrt[3]{y^2}$

Ř: $\frac{dy}{dx} = 3 \cdot \sqrt[3]{y^2}$!

$$\frac{1}{3} \int \frac{dy}{y^{\frac{2}{3}}} = \int dx$$

zde přicházíme o nulové řešení

$$\frac{1}{3} \frac{y^{\frac{1}{3}}}{\frac{1}{3}} = x + C$$

$$y^{\frac{1}{3}} = x + C$$

$$y = (x+C)^3, \; C \in \mathbb{R}$$

Nulové řešení je třeba přidat. Nezískáme ho žádnou volbou C.

$$\underline{y = (x+C)^3, \; C \in \mathbb{R} \quad a \quad y = 0}$$

5. $\frac{y'}{y} = \frac{1}{x-1}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1}$$

$$\int \frac{1}{y}dy = \int \frac{1}{x-1}\,dx$$

$$\ln|y| = \ln|x-1| + c$$

$$e^{\ln|y|} = e^{\ln|x-1|+c}$$

$$|y| = e^{c}\cdot e^{\ln|x-1|}$$

$$y = \pm e^{c}\cdot(x-1), \quad x \neq 1, \quad k = \pm e^{c}$$

$$\underline{y = k(x-1), \quad k \neq 0, x \neq 1}$$

6. $y^2 y' = \cos x$

$$y^2 \frac{dy}{dx} = \cos x$$

$$\int y^2 dy = \int \cos x\,dx$$

$$\frac{y^3}{3} = \sin x + C$$

$$y^3 = 3\sin x + k$$

$$\underline{y = \sqrt[3]{3\sin x + k} \qquad x \in \mathbb{R},\ k \in \mathbb{R}}$$

7. $e^y \cdot y' = 1$

$\int e^y\,dy = \int dx$

$e^y = x + C$

$\ln e^y = \ln(x+C)$

$\underline{y = \ln(x+C), \; x > -C, \; C \in \mathbb{R}}$

8. $y' = y$

$\frac{dy}{dx} = y$

$\int \frac{1}{y}\,dy = \int dx$

$\ln|y| = x + C$

$y = \pm e^{x+C}$

$y = \pm e^{C} \cdot e^{x} \qquad k = \pm e^{C}$

$\underline{y = k \cdot e^{x} \quad k \in \mathbb{R}}$

Singulární řešení získáme volbou $k = 0$.

9. $\frac{1}{y}\,y' = -2$

$\frac{1}{y}\frac{dy}{dx} = -2$

$\int \frac{1}{y}\,dy = -2\int dx$

$\ln|y| = -2x + C$

$y = \pm e^{-2x+C}$

$y = \pm e^{C} \cdot e^{-2x}$

$y = k \cdot e^{-2x}$

$x \in \mathbb{R}, \; \underline{k \neq 0}$!

Případ $y = 0$ vylučuje hned zadání.

10. $xy' - y = 0$

$$x\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{x} \quad , x \neq 0 \text{, přidáme o singul. řešení}$$

$$\int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\ln|y| = \ln|x| + \ln C$$

$$\ln|y| = \ln|Cx|$$

$$|y| = Cx$$

$$y = \pm Cx \qquad x\neq 0,\ C\neq 0$$

volbou nové konstanty

$$\underline{y = kx} \qquad k = \pm C$$

a) Případ $x=0$ ověříme dosazením

b) Pro $k=0$ obdržíme $y=0$

11. $\dfrac{y'}{4\sqrt{y}} = 6x^2 \qquad y>0 \qquad 4x^3+2C>0$

$$4x^3 > -2C$$

$$\frac{1}{4}\int y^{-\frac{1}{2}}dy = 6\int x^2 dx \qquad x^3 > -\frac{2C}{4}$$

$$\frac{1}{2}\sqrt{y} = 2x^3 + C \qquad x > \sqrt[3]{-\frac{C}{2}}$$

$$\sqrt{y} = 4x^3 + 2C$$

$$\underline{y = (4x^3+2C)^2}\ , \ x\in\left(\sqrt[3]{-\frac{C}{2}}, \infty\right)$$

$$C\in\mathbb{R}$$

12.

$$\frac{1}{y+1}\,y' = \cotg x \qquad y \neq -1$$

$$\frac{1}{y+1}\,\frac{dy}{dx} = \cotg x$$

$$\int \frac{dy}{y+1} = \int \cotg x \, dx$$

$$\ln|y+1| = \ln|\sin x| + \ln|C|$$

$$|y+1| = C \sin x$$

$$y+1 = \pm C \sin x \qquad k = \pm C$$

$$y = k \sin x - 1, \quad x \neq k\pi, \; k \in \mathbb{Z}, \; k \neq 0$$

Domácí cvičení

1. $3y^2 y' = 2\cos\frac{x}{2}$ V: $y = \sqrt[3]{\sin\frac{x}{2} + C}$, $x \in \mathbb{R}$, $C \in \mathbb{R}$

2. $\frac{1}{y-3}\,y' = 6x^2$ V: $y = k \cdot e^{2x^3} + 3$, $x \in \mathbb{R}$, $k \neq 0$

3. $\frac{1}{y}\,y' = -4$ V: $y = k \cdot e^{-4x}$, $x \in \mathbb{R}$, $k \neq 0$

4. $\cancel{y'}$

Homogenní DR:

POSTUP

$$F(x, y, y') = 0 \xrightarrow[\text{LZE PŘEVÉST}]{} y' = f\left(\frac{y}{x}\right)$$

SUBSTITUCE $y = ux \rightarrow$ LZE PŘEVÉST NA DR se separ. prom.

! nezapomeňte nahradit y':

$$y = ux \Rightarrow y' = u'x + u$$

$$y' = f\left(\frac{y}{x}\right)$$

$$u'x + u = f(u)$$

$$x\frac{du}{dx} = f(u) - u$$

$$\frac{du}{f(u)-u} = \frac{dx}{x} \Rightarrow u = g(x, C)$$

$$\frac{y}{x} = g(x, C)$$

$$\underline{y = x \cdot g(x, C)}$$

43

1. $xy' = 2x + y$

Ř $y' = 2 + \frac{y}{x}$

substituce $u = \frac{y}{x}$ $(u'x + u = y')$

$$u'x + u = 2 + u$$

$$u' = \frac{2}{x}$$

$$\frac{du}{dx} = \frac{2}{x}$$

$$\int du = 2\int \frac{1}{x}dx = 2\ln|x|$$

$$u = 2\ln|x| + \ln C$$

$$u = 2\ln|Cx|$$

$$\frac{y}{x} = 2\ln|Cx|$$

$$\underline{y = 2x\ln|Cx|}$$

$$\underline{x \neq 0, C \neq 0}$$

2. $xy' = y \ln \frac{y}{x}$ HOM. DR.

Ř: $y' = \frac{y}{x} \ln \frac{y}{x}$

$$u'x + u = u \ln u$$
$$u'x = u \ln u - u$$
$$u' = \frac{u(\ln u - 1)}{x}$$
$$\frac{u'}{u(\ln u - 1)} = \frac{1}{x}$$
$$\frac{1}{u(\ln u - 1)} \frac{du}{dx} = \frac{1}{x}$$
$$\int \frac{du}{u(\ln u - 1)} du = \int \frac{1}{x} dx \qquad u \neq 0,\ u \neq e$$

substituce $t = \ln u - 1$
$dt = \frac{1}{u} du$

$$\int \frac{1}{t} dt = \int \frac{1}{x} dx$$
$$\ln|t| = \ln|x| + \ln(c)$$
$$\ln|t| = \ln|cx|$$
$$|t| = cx$$
$$t = \pm cx \qquad k = \pm c$$
$$\ln u - 1 = kx$$
$$\ln u = kx + 1$$
$$u = e^{kx+1}$$
$$\frac{y}{x} = e^{kx+1}$$
$$y = xe^{kx+1}, \quad y = ex, \quad y = 0$$

celkem
$\underline{y = xe^{kx+1}}$
$\underline{x \in \mathbb{R} - \{0\}}$

$$y' = \frac{x}{y}$$ HOM. RCE

Ř: $y' = \dfrac{1}{\frac{y}{x}}$ $\quad u = \dfrac{y}{x}$

$$u'x + u = \frac{1}{u}$$

$$u'x = \frac{1}{u} - u$$

$$u' = \frac{\frac{1-u^2}{u}}{x} = \frac{1-u^2}{u} \cdot \frac{1}{x}$$

$$\frac{u}{1-u^2} u' = \frac{1}{x}$$

$$\int \frac{u}{1-u^2} du = \int \frac{1}{x} dx$$

substituce
$t = 1-u^2$
$dt = -2u\,du$
$\dfrac{dt}{-2} = u\,du$

$$-\frac{1}{2} \int \frac{1}{t} dt = \int \frac{1}{x} dx$$

$$-\frac{1}{2} \ln|t| = \ln|x| + \ln|c|$$

$$\ln|t| = -2 \ln|cx|$$

$$\ln|t| = \ln \frac{1}{c^2x^2}$$

$$t = \frac{1}{c^2x^2}$$ zpětné dosazení

$$1-u^2 = \frac{1}{c^2x^2}$$

$$u^2 = 1 - \frac{1}{c^2x^2}$$

$x^2 - \dfrac{1}{c^2} \geq 0$

$|x| \geq \dfrac{1}{c}$, $c \neq 0$

$$\frac{y^2}{x^2} = 1 - \frac{1}{c^2x^2}$$

$$y^2 = x^2 - \frac{1}{c^2}$$

$$\underline{y = \pm\sqrt{x^2 - \frac{1}{c^2}}}$$

$u = \pm 1 \Rightarrow$ $\underline{y = \pm x}$

Homogenní DR
Domácí cvičení

1. $x^2 + y^2 = 2xyy'$

V: $y = \pm\sqrt{x^2 - \frac{x}{c}}$ a $y = \pm x$, $x \in \mathbb{R} - \{0\}$
$x^2 - \frac{x}{c} \geq 0$, $c \neq 0$

2. $yy' = 2y - x$

V: $e^{\frac{x}{y-x}} = k(y-x)$ a $y = x$

3. $xy' \cos\frac{y}{x} = y \cos\frac{y}{x} - x$

V: $\sin\frac{y}{x} = \ln\frac{c}{x}$, $x \in \mathbb{R} - \{0\}$, $\frac{c}{x} > 0$, $c \neq 0$

Lineární diferenciální rovnice 1. řádu

$$y' + p(x) \cdot y = q(x)$$

a) Lagrangeova metoda variace konstant

1. $y' + p(x)y = 0$

$$y^* = C e^{-\int p(x)\,dx}$$ zkrácené řeš. zhomogeniz. rovnice

2. obecné řešení hledáme ve tvaru

$y = C(x) \cdot e^{-\int p(x)\,dx}$, $C(x)$ ... neznámá fce

Pak

$$y' = C'(x) \cdot e^{-\int p(x)\,dx} + C(x) \cdot e^{-\int p(x)\,dx} (-p(x))$$ (zderivujeme)

dosadíme do zadané rovnice $y' + p(x)y = q(x)$

$$\underbrace{C'(x) \cdot e^{-\int p(x)\,dx} - C(x) \cdot e^{-\int p(x)\,dx} \cdot p(x)}_{y'} + p(x)\Big(\underbrace{C(x) \cdot e^{-\int p(x)\,dx}}_{y}\Big) = q(x)$$

$$C'(x) \cdot e^{-\int p(x)\,dx} = q(x)$$

$$C'(x) = q(x) \cdot e^{\int p(x)\,dx}$$

$$C(x) = \int q(x) \cdot e^{\int p(x)\,dx} + k$$

$$y = \left(\int q(x) \cdot e^{\int p(x)\,dx} + k\right) \cdot e^{-\int p(x)\,dx}$$

b) Jiný postup: Bernoulliova substituce

Předp. že $y' + p(x)y = 0$ má řeš. tvaru $y(x) = u(x) \cdot v(x)$

Pak $y' = u'v + uv'$. Dosadíme do zadání

$$u'v + uv' + p(x)\,uv = q(x)$$

$$u'v + u[v' + v \cdot p(x)] = q(x)$$ Zvolíme $v' + v \cdot p(x) = 0$

(*) $v = e^{-\int p(x)\,dx}$ dosadíme do rovnice

$y' \cdot tg\, x - y = 1$ $\qquad x \neq (2k+1)\frac{\pi}{2}$

LDR

Ř: I. $y' \cdot tg\, x - y = 0$

$$y' = \frac{y}{tg\, x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$\int \frac{dy}{y} = \int \frac{\cos x}{\sin x}\, dx$$

$$\ln|y| = \ln|\sin x| + \ln C$$

$$y^* = C \sin x$$

II. $y = C(x) \sin x$

$$y' = C'(x) \sin x + C(x) \cos x$$

dosadíme do zadání

$$\left[C'(x)\sin x + C(x)\cos x\right] \cdot \frac{\sin x}{\cos x} - C(x) \sin x = 1$$

$$C'(x) \frac{\sin^2 x}{\cos x} = 1$$

$$C'(x) = \frac{\cos x}{\sin^2 x}$$

$$C(x) = \int \frac{\cos x}{\sin^2 x}\, dx = -\frac{1}{\sin x} + k$$

celkem

$$y = C(x) \cdot \sin x = \left(-\frac{1}{\sin x} + k\right) \sin x = k \sin x - 1$$

$$k \neq (2k+1)\frac{\pi}{2}, \; k \in R$$

$$y' - xy = e^{\frac{1}{2}x(x+2)}$$

LDR 1. řádu
lineární dif. rce

Ř: Zhomogenizovaná DR

Ⓘ $y' - xy = 0$

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\frac{dy}{dx} = x$$

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + c$$

$$y^* = \bar{C} e^{\frac{x^2}{2}}$$

↗ obecné řešení zhomog. rce

Ⓘⓘ $y = C(x)\cdot e^{\frac{x^2}{2}}$ předpokládaný tvar

$$y' = C'(x)e^{\frac{x^2}{2}} + C(x)\cdot e^{\frac{x^2}{2}}\cdot \frac{1}{2}2x$$

dosadíme do zadání

$$\left[C'(x)\cdot e^{\frac{x^2}{2}} + xC(x)e^{\frac{x^2}{2}}\right] - x\cdot C(x)e^{\frac{x^2}{2}} = e^{\frac{x(x+2)}{2}}$$

$$C'(x)\cdot e^{\frac{x^2}{2}} = e^{\frac{x^2+2x}{2}} \quad \Big/ e^{-\frac{x^2}{2}}$$

$$C'(x) = e^{\frac{x^2}{2} + x - \frac{x^2}{2}}$$

$$C'(x) = e^x$$

$$C(x) = \int e^x\,dx = e^x + k$$

Celkem:

$$\underline{y = (e^x + k)\cdot e^{\frac{x^2}{2}}} \qquad \underline{x \in \mathbb{R},\ k \in \mathbb{R}}$$

Samostatné cvičenie

$$xy' + y + y^2 = 0$$

$$xy' = -y - y^2$$

$$y' = -\frac{y+y^2}{x}$$

$$\frac{y'}{y(1+y)} = -\frac{1}{x} \qquad y \neq 0,\ y \neq -1,\ x \neq 0$$

$$\frac{1}{y(1+y)} \cancel{\frac{dy}{dx}} \cancel{=}\, dy = -\frac{1}{x}\,dx$$

Rozklad na P.Z

$$\frac{1}{y(1+y)} = \frac{A}{y} + \frac{B}{y+1} = \frac{1}{y} + \frac{-1}{y+1}$$

$$1 = A(y+1) + By \qquad \begin{array}{l} y=0 \Rightarrow A=1 \\ y=-1 \Rightarrow B=-1 \end{array}$$

$$\int \frac{1}{y}\,dy - \int \frac{1}{y+1}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| - \ln|y+1| = -\ln|x| + \ln|C|$$

$$\ln\left|\frac{y}{y+1}\right| = \ln\left|\frac{C}{x}\right|$$

$$\left|\frac{y}{y+1}\right| = \frac{C}{x}$$

$$\frac{y}{1+y} = \pm\frac{C}{x} \qquad k = \pm C$$

$$\frac{y}{1+y} = \frac{k}{x}$$

$$y = \frac{k}{x}(1+y) = \frac{k}{x} + \frac{k}{x}y$$

$$y - \frac{k}{x}y = \frac{k}{x}$$

$$y\left(1-\frac{k}{x}\right) = \frac{k}{x}$$

$$\underline{y = \frac{k}{x\left(1-\frac{k}{x}\right)} = \frac{k}{x-k}} \quad \text{a} \quad \underline{y=-1}$$

... $y=0$ získame voľbou $k=0$, $y=-1$ je treba pripísať zvlášť

Samostatné cvičení

$$xy' - \frac{y}{x+1} = 0 \qquad x \neq -1$$

$$xy' = \frac{y}{x+1}$$

$$\frac{y'}{y} = \frac{1}{x(x+1)} \qquad x \neq 0,\ x \neq -1$$

Rozklad na P. Z.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

$$x = 0 \Rightarrow A = 1$$

$$x = -1 \Rightarrow B = -1$$

$$\frac{y'}{y} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$\ln|y| = \ln|x| - \ln|x+1| + \ln|C|$$

$$\ln|y| = \ln\left|\frac{Cx}{x+1}\right|$$

$$|y| = \frac{Cx}{x+1}$$

$$y = \pm \frac{Cx}{x+1}$$

Pro $k=0$ získáme $y=0$

$x \in \mathbb{R} - \{-1\},\ k \in \mathbb{R}$

LDR

$$y' + 2y = 4x$$

Ř: Bernoulliova substituce:

$y = uv$

$y' = u'v + uv'$

$$u'v + uv' + 2uv = 4x$$

$$u'v + u(v' + 2v) = 4x$$

volitelná podmínka: $\Rightarrow v' + 2v = 0$ tj.

$$v' = -2v$$

$$\frac{dv}{dx} = -2v$$

$$\frac{dv}{v} = -2dx$$

$$\int \frac{1}{v} dv = -2\int dx$$

$$\ln|v| = -2x$$

$$v = e^{-2x}$$

$$u'v = 4x$$

$$\cancel{\frac{du}{dx} v = 4x}$$

$$u' \cdot e^{-2x} = 4x$$

$$u' = 4xe^{2x}$$

$$u = \int 4xe^{2x}\, dx$$

$$= \begin{vmatrix} f = 4x & f' = 4 \\ g' = e^{2x} & g = \frac{1}{2}e^{2x} \end{vmatrix} = 2xe^{2x} - \int 2e^{2x}\, dx =$$

$$= 2xe^{2x} - e^{2x} + C$$

$$y = (\underbrace{2xe^{2x} - e^{2x} + C}_{u}) \cdot \underbrace{(e^{-2x})}_{v} = 2x - 1 + Ce^{-2x}$$

$$\underline{y = 2x - 1 + Ce^{-2x}}, \quad x \in \mathbb{R},\ c \in \mathbb{R}$$

Samostatné cvičení HOM. DR.

$$xy' + y = x$$

Ř: $xy' = x - y$

$$y' = \frac{x-y}{x}$$

$$y' = 1 - \frac{y}{x}$$

Substituce $u = \frac{y}{x}$

$$u'x + u = 1 - u$$

$$u'x = 1 - 2u$$

$$u' = \frac{1-2u}{x}$$

$$\frac{1}{1-2u}\frac{du}{dx} = \frac{1}{x}$$

$$\int \frac{1}{1-2u}\,du = \int \frac{1}{x}\,dx$$

sub. $t = 1 - 2u$
$dt = -2du$
$du = \frac{dt}{-2}$

$$-\frac{1}{2}\cdot\int \frac{dt}{t} = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|t| = \ln|x| + \ln|C|$$

$$\ln|t| = -2\ln|Cx|$$

$$\ln|t| = \ln\frac{1}{c^2x^2}$$

$$t = \frac{1}{c^2x^2}$$

$$1 - 2u = \frac{1}{c^2x^2}$$

$$2u = 1 - \frac{1}{c^2x^2} = \frac{c^2x^2-1}{c^2x^2}$$

$$\frac{y}{x} = \frac{c^2x^2-1}{2c^2x^2}$$

$u = \frac{1}{2}$
$\Downarrow$

$x \neq 0,\ c \neq 0$ $\quad y = \frac{c^2x^2-1}{2c^2\cdot x},\quad y = \frac{1}{2}x$

$$x^2 y' - 2xy = -3$$

Bernoulli: $y = uv$

$y' = u'v + uv'$

$$x^2(u'v + uv') - 2xuv = -3$$

$$x^2 u'v + u(\underbrace{x^2v' - 2xv}_{=0}) = -3$$

volitelná podmínka $x^2v' - 2xv = 0$ $\Downarrow$

$$xv' = 2v$$

$$x\frac{dv}{dx} = 2v$$

$$\frac{dv}{v} = 2\frac{1}{x}dx$$

$$\int\frac{dv}{v} = 2\int\frac{1}{x}dx$$

$$\ln|v| = 2\ln x$$

$$\ln v = \cancel{2}\ln x^2$$

$$v = x^2$$

$$x^2 \cdot u' \cdot x^2 = -3$$

$$u' = \frac{-3}{x^4}$$

$$\int du = \int\frac{-3}{x^4}dx$$

$$u = \frac{1}{x^3} + c$$

celkem

$$y = uv = \left(\frac{1}{x^3} + c\right)x^2 = \frac{1}{x} + Cx^2$$

$$y = \frac{1}{x} + Cx^2$$