

# Quantum Computing in complex Geometric Algebras

Propaganda of split signature

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2nd (online) GA meeting



# The role of QRA

Jaroslav Hrdina, Dietmar Hildenbrand, Aleš Návrát, Christian Steinmetz, Rafael Alves, Carlile Campos Lavor, Petr Vašík, Ivan Eryganov, *Quantum Register Algebra: the mathematical language for quantum computing*. Quantum Inf Process 22, 328 (2023)

$n$  – qubit system

$$\mathbb{G}_{2n+2} = \langle e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n, r_1, r_2 \rangle$$

New (Witt) basis

$$f_i = \frac{1}{2}(e_i + \iota \bar{e}_i), \quad f_i^\dagger = \frac{1}{2}(e_i - \iota \bar{e}_i), \quad \iota = r_1 r_2$$

$$|a_1 \cdots a_n\rangle \hookrightarrow (f_1^\dagger)^{a_1} \cdots (f_n^\dagger)^{a_n} I, \quad a_i \in \{0, 1\}, \quad I = f_1 f_1^\dagger \cdots f_n f_n^\dagger$$

$$\langle a_n \cdots a_1| \hookrightarrow I (f_1)^{a_1} \cdots (f_n)^{a_n} I, \quad a_i \in \{0, 1\}$$

For example: X - gate:  $|0\rangle\langle 1| + |1\rangle\langle 0| = f^\dagger + f$

and  $(|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle = (f^\dagger + f)f^\dagger I = I$

# The role of QRA

$$\text{QRA: } \mathbb{G}_{2n+2} = \langle e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n, r_1, r_2 \rangle$$

$$f_i = \frac{1}{2}(e_i + \iota \bar{e}_i), \quad f_i^\dagger = \frac{1}{2}(e_i - \iota \bar{e}_i), \quad \iota = r_1 r_2$$

$$\text{Fact: } \iota^2 = (r_1 r_2)^2 = -1, \iota \mathbb{G}_{2n} = \mathbb{G}_{2n} \iota, \quad \text{QRA} = \mathbb{G}_{2n} \otimes \mathbb{G}_2^+$$

$$\mathbb{C} \mathbb{G}_{2n} \cong \mathbb{G}_{2n} \otimes \mathbb{C} = \langle e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n \rangle$$

$$f_i = \frac{1}{2}(e_i + \iota \bar{e}_i), \quad f_i^\dagger = \frac{1}{2}(e_i - \iota \bar{e}_i), \quad \iota \in \mathbb{C}$$

$$\boxed{\text{QRA} \cong \mathbb{G}_{2n} \otimes \mathbb{G}_2^+ \cong \mathbb{G}_{2n} \otimes \mathbb{C}}$$

$$Q(x) = x_1^2 + x_2^2 + \cdots + x_n^2, \text{ basis } e_1, e_2, \dots, e_n$$

$$Q(x) = x_1^2 - x_2^2 + \cdots + x_n^2, \text{ basis } e_1, \iota e_2, \dots, e_n$$

$$\mathbb{C}\mathbb{G}_{n,n} = \mathbb{G}_{n,n} \otimes \mathbb{C} = \langle e_1, \dots, e_n, \iota \bar{e}_1, \dots, \iota \bar{e}_n \rangle$$

$$f_i = \frac{1}{2}(e_i + \tilde{e}_i), \quad f_i^\dagger = \frac{1}{2}(e_i - \tilde{e}_i), \quad \iota \in \mathbb{C}$$

split-QRA:  $\mathbb{G}_{n+2,n} = \langle e_1, \dots, e_n, r_1, r_2, \bar{e}_1, \dots, \bar{e}_n \rangle$

$$f_i = \frac{1}{2}(e_i + \bar{e}_i), \quad f_i^\dagger = \frac{1}{2}(e_i - \bar{e}_i), \quad \iota = r_1 r_2$$

1-qubit system  $\mathbb{C}\mathbb{G}_{1,1}$

$$|0\rangle := (1)ff^\dagger = ff^\dagger, \quad |1\rangle := (f^\dagger)ff^\dagger = f^\dagger.$$

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| = f^\dagger + f,$$

$$\sigma_y = i(|0\rangle\langle 1| - |1\rangle\langle 0|) = i(f^\dagger - f),$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| = ff^\dagger - f^\dagger f,$$

# Matrix representation

$$Cl(p+1, q+1) = M_2(Cl(p, q))$$

In the case of  $\mathbb{G}_{1,1} = M_2(Cl(0,0)) = M_2(\mathbb{C})$ .

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_- \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_+ e_- \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and in this formalism, we can derive Witt pairs as

$$f = \frac{1}{2} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$f^\dagger = \frac{1}{2} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where  $I = ff^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\alpha|0\rangle + \beta|1\rangle = \alpha I + \beta f^\dagger \Leftrightarrow \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} \cong \mathbb{C}^2$$

# Pauli gates

1-qubit system  $\mathbb{C}^2$  The Pauli gates can be expressed as matrices

$$\sigma_x = f^\dagger + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = if^\dagger - if = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_z = ff^\dagger - f^\dagger f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

Multiplications preserve the columns:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix}$$

By iterating on the process, we receive  $\mathbb{C}\mathbb{G}_{2,2}$  as  $4 \times 4$  block matrices over  $\mathbb{C}$  or  $2 \times 2$  block matrices over  $\mathbb{G}\mathbb{G}_{1,1}$

$$1 \mapsto \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, e_+ \mapsto \begin{pmatrix} e_+ & 0 \\ 0 & -e_+ \end{pmatrix}, e_- \mapsto \begin{pmatrix} e_- & 0 \\ 0 & -e_- \end{pmatrix},$$

$$g_+ \mapsto \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, g_- \mapsto \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

where  $\mathbf{1}$  is  $2 \times 2$  identity matrix or we can see  $\{\mathbf{1}, e_+, e_-, e_+e_-\}$  as a generators of split quaternion algebra.



Now,

$$f_1 = \frac{1}{2} \left( \begin{pmatrix} e_+ & 0 \\ 0 & -e_+ \end{pmatrix} + \begin{pmatrix} e_+ & 0 \\ 0 & -e_+ \end{pmatrix} \right) = \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix},$$

$$f_1^\dagger = \frac{1}{2} \left( \begin{pmatrix} e_+ & 0 \\ 0 & -e_+ \end{pmatrix} + \begin{pmatrix} e_+ & 0 \\ 0 & -e_- \end{pmatrix} \right) = \begin{pmatrix} f^\dagger & 0 \\ 0 & -f^\dagger \end{pmatrix},$$

$$f_2 = \frac{1}{2} \left( \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix},$$

$$f_2^\dagger = \frac{1}{2} \left( \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix},$$

$$I = \begin{pmatrix} ff^\dagger & 0 \\ 0 & ff^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & ff^\dagger \end{pmatrix},$$

$$f_1^\dagger f_2^\dagger = \begin{pmatrix} f^\dagger & 0 \\ 0 & -f^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & f^\dagger \\ 0 & 0 \end{pmatrix}.$$

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{10}f^\dagger & \alpha_{01} + \alpha_{11}f^\dagger \\ 0 & \alpha_{00} - \alpha_{10}f^\dagger \end{pmatrix} ff^\dagger \cong \mathbb{C}^4$$

## Parallel gate example: $\sigma_x \otimes \sigma_y$

$$\begin{aligned}\sigma_x \otimes \sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes i(f^\dagger - f) = if_1^\dagger f_2^\dagger - if_1^\dagger f_2 - if_1 f_2^\dagger + if_1 f_2 \\&= i \begin{pmatrix} f^\dagger & 0 \\ 0 & -f^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} f^\dagger & 0 \\ 0 & -f^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix} \\&\quad - i \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix} \\&= i \begin{pmatrix} 0 & f^\dagger \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} 0 & 0 \\ -f^\dagger & 0 \end{pmatrix} - i \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 0 \\ f^\dagger - f & 0 \end{pmatrix} \\&= i \begin{pmatrix} 0 & f^\dagger - f \\ f^\dagger - f & 0 \end{pmatrix}\end{aligned}$$

## Parallel gate example: $\sigma_y \otimes \sigma_x$

$$\sigma_y \otimes \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes (f^\dagger + f) = \begin{pmatrix} 0 & -i(f^\dagger + f) \\ i(f^\dagger + f) & 0 \end{pmatrix}$$

Thank you for your attention!