POČET INTEGRÁLNÍ – CVIČENÍ 1M

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OBSAH:

Část I: Substituční metoda

1. $$\int (3x+5)^{73}\,dx = \left| \begin{array}{l} t = 3x+5 \\ dt = 3dx \\ dx = \frac{dt}{3} \end{array} \right| = \int t^{73} \cdot \frac{dt}{3} =$$

$$= \frac{1}{3}\int t^{73}\,dt = \frac{1}{3} \cdot \frac{t^{74}}{74} = \frac{1}{222} \cdot (3x+5)^{74} + C$$

2. $$\int \sqrt[8]{2x+5}\,dx = \int (2x+5)^{\frac{1}{8}}\,dx = \left| \begin{array}{l} t = 2x+5 \\ dt = 2dx \\ dx = \frac{dt}{2} \end{array} \right|$$

$$= \int t^{\frac{1}{8}} \cdot \frac{dt}{2} = \frac{1}{2}\int t^{\frac{1}{8}}\,dt = \frac{1}{2}\, \frac{t^{\frac{1}{8}+1}}{\frac{9}{8}} =$$

$$= \frac{1}{2} \cdot \frac{8}{9} \cdot t^{\frac{9}{8}} = \frac{4}{9}\sqrt[8]{t^9} = \frac{4}{9}\sqrt[8]{(2x+5)^9} + C$$

3. $$\int \frac{\sin x}{1+3\cos x}\,dx = \left| \begin{array}{l} t = 1+3\cos x \\ dt = 3(-\sin x)dx \\ \frac{dt}{-3} = \sin x\,dx \end{array} \right| = \int \frac{1}{t} \cdot \frac{dt}{-3}$$

$$= -\frac{1}{3}\int \frac{1}{t}\,dt = -\frac{1}{3}\ln|t| = -\frac{1}{3}\ln|1+3\cos x| + C$$

4. $$\int \frac{e^x\sqrt{\operatorname{arctg} e^x}}{1+e^{2x}}\,dx = \left| \begin{array}{l} t = e^x \\ dt = e^x\,dx \end{array} \right| = \int \frac{\sqrt{\operatorname{arctg} t}}{1+t^2}\,dt$$

$$= \left| \begin{array}{l} u = \operatorname{arctg} t \\ du = \frac{1}{1+t^2}\,dt \end{array} \right| = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} =$$

$$= \frac{2}{3}\sqrt{u^3} = \frac{2}{3}\sqrt{\operatorname{arctg}^3 t} = \frac{2}{3}\sqrt{\operatorname{arctg}^3 e^x} + C$$

5. $\int x^3\sqrt{5x^2+3}\,dx = \left| \begin{array}{ll} t = 5x^2+3 & \frac{dt}{10} = x\,dx \\ dt = 10x\,dx & t-3 = 5x^2 \\ \frac{t-3}{5} = x^2 \Rightarrow x^3 dx = x^2 \cdot x\,dx = \frac{t-3}{5}\cdot\frac{dt}{10} = \frac{t-3}{50}dt & \end{array} \right|$

$$= \int \sqrt{t}\cdot \frac{t-3}{50}\,dt = \frac{1}{50}\int t^{\frac{1}{2}}(t-3)\,dt$$

$$= \frac{1}{50}\int t^{\frac{3}{2}} - 3t^{\frac{1}{2}}\,dt = \frac{1}{50}\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 3\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) =$$

$$= \frac{1}{50}\left(\frac{2}{5}\sqrt{t^5} - 3\cdot\frac{2}{3}\sqrt{t^3}\right) = \frac{1}{125}\sqrt{t^5} - \frac{1}{25}\sqrt{t^3} =$$

$$= \frac{1}{125}\sqrt{(5x^2+3)^5} - \frac{1}{25}\sqrt{(5x^2+3)^3} + C$$

6. $\int \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \right| =$

$$\int \sqrt{1-\sin^2 t}\cdot \cos t\,dt =$$

VZOREC $\sin^2 t + \cos^2 t = 1 \Rightarrow 1 - \sin^2 t = \cos^2 t$

$$\int \sqrt{\cos^2 t}\cdot \cos t\,dt \text{ ,,=''} \int \cos^2 t\,dt =$$

$$\int \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(\int dt + \int \cos 2t\,dt\right) =$$

$$= \frac{1}{2}t + \frac{1}{4}\sin 2t = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + C$$

Pomocné výpočty následují:

Platí:

$\sin^2 t + \cos^2 t = 1$

$\cos^2 t - \sin^2 t = \cos 2t$

$2\cos^2 t = 1 + \cos 2t$

Podobně:

⋮

$$\boxed{\cos^2 t = \frac{1+\cos 2t}{2}, \qquad \sin^2 t = \frac{1-\cos 2t}{2}}$$

$$\int \cos 2t \, dt = \left| \begin{array}{l} u = 2t \\ du = 2dt \\ \underline{dt = \frac{du}{2}} \end{array} \right| = \int \cos u \, \frac{du}{2} =$$

$$= \frac{1}{2}\int \cos u \, du = \frac{1}{2}\sin u = \underline{\underline{\frac{1}{2}\sin 2t}}$$

Domácí cvičení: Vypočtěte pomocí substituční metody integrály:

1. $\int (1-x)^5 \, dx$, $\quad V = -\frac{1}{6}(1-x)^6 + C$
2. $\int \sqrt{2x-1} \, dx$, $\quad V = \frac{1}{3}\sqrt{(2x-1)^3} + C$
3. $\int \frac{x^2}{(1-x^3)^2} \, dx$, $\quad V = \frac{1}{3}\,\frac{1}{1-x^3} + C$
4. $\int \frac{\sqrt{1+\ln x}}{x} \, dx$, $\quad V = \frac{2}{3}\sqrt{(1+\ln x)^3} + C$
5. $\int \sin^2 x \, dx$, $\quad V = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Část II: Per partes

1. $\int x \sin x \, dx = \begin{vmatrix} f = x & f' = 1 \\ g' = \sin x & g = -\cos x \end{vmatrix} =$

$= x(-\cos x) - \int 1 \cdot (-\cos x)\, dx = -x\cos x + \int \cos x \, dx$

$= \underline{-x\cos x + \sin x + C}$

2. $\int x^2 e^x \, dx = \begin{vmatrix} f = x^2 & f' = 2x \\ g' = e^x & g = e^x \end{vmatrix} =$

$= x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2\int x e^x \, dx =$

$\begin{vmatrix} f = x & f' = 1 \\ g' = e^x & g = e^x \end{vmatrix} = x^2 e^x - 2\left(x e^x - \int 1 \cdot e^x \, dx\right) =$

$= x^2 e^x - 2x e^x + 2\int e^x \, dx = \underline{x^2 e^x - 2x e^x + 2e^x + C}$

3. $A = \int \sin^2 x \, dx = \begin{vmatrix} f = \sin x & f' = \cos x \\ g' = \sin x & g = -\cos x \end{vmatrix} =$

$= -\sin x \cos x - \int \cos x (-\cos x)\, dx =$

$= -\sin x \cos x + \int \cos^2 x \, dx =$

$= -\sin x \cos x + \int 1 - \sin^2 x \, dx =$

$= -\sin x \cos x + \int dx - \int \sin^2 x \, dx$

$= -\sin x \cos x + x - A$

Rovnice $A = -\sin x \cos x + x - A$

$2A = x - \sin x \cos x$

$\underline{A = \frac{1}{2}(x - \sin x \cos x) + C}$

4. $A = \int \frac{\operatorname{arctg} x}{1+x^2}\,dx = \left| \begin{array}{ll} f = \operatorname{arctg} x & f' = \frac{1}{1+x^2} \\ g' = \frac{1}{1+x^2} & g = \operatorname{arctg} x \end{array} \right|$

$= \operatorname{arctg}^2 x - \int \frac{\operatorname{arctg} x}{1+x^2}\,dx$

Rovnice $\quad A = \operatorname{arctg}^2 x - A$

$2A = \operatorname{arctg}^2 x$

$\underline{A = \frac{1}{2}\operatorname{arctg}^2 x + C}$

5. $\int \ln x\,dx = \left| \begin{array}{ll} f = \ln x & f' = \frac{1}{x} \\ g' = 1 & g = x \end{array} \right| =$

$= x \cdot \ln x - \int \frac{1}{x} \cdot x\,dx = x \ln x - \int dx =$

$= \underline{x \ln x - x + C}$

6. $\int \frac{\ln^2 x}{\sqrt{x}}\,dx = \left| \begin{array}{ll} f = \ln^2 x & f' = 2\ln x \cdot \frac{1}{x} \\ g' = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} & g = 2\sqrt{x} \end{array} \right|$

$= 2\sqrt{x}\ln^2 x - \int 2\ln x \cdot \frac{1}{x} \cdot 2\sqrt{x}\,dx =$

$= 2\sqrt{x}\ln^2 x - 4\int \frac{\ln x}{\sqrt{x}}\,dx = \left| \begin{array}{ll} f = \ln x & f' = \frac{1}{x} \\ g = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} & g' = 2\sqrt{x} \end{array} \right| =$

$= 2\sqrt{x}\ln^2 x - 4\left(2\sqrt{x}\ln x - \int \frac{1}{x} \cdot 2\sqrt{x}\,dx\right) =$

$= 2\sqrt{x}\ln^2 x - 8\sqrt{x}\ln x + 8\int x^{-\frac{1}{2}}\,dx =$

$= \underline{2\sqrt{x}\ln^2 x - 8\sqrt{x}\ln x + 16\sqrt{x} + C}$

Domácí cvičení: Vypočtěte pomocí metody per-partes integrály

1. $\int x \ln^2 x \, dx$, $V = \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + c$

2. $\int \cos(\ln x) \, dx$, $V = \frac{1}{2}(x \cos(\ln x) + x \sin \ln(x)) + C$

3. $\int \ln^3 x \, dx$, $V = x(\ln^3 x - 3\ln^2 x + 6\ln x - 6) + C$

Část III. Kombinace substituce a per partes

1. $$\int x e^{2x} dx = \left| \begin{array}{l} t = 2x \\ dt = 2dx \\ dx = \frac{dt}{2} \end{array} \right| = \int \frac{t}{2} \cdot e^t \frac{dt}{2} =$$

$$= \frac{1}{4} \int t e^t dt = \left| \begin{array}{ll} f = t & f' = 1 \\ g' = e^t & g = e^t \end{array} \right| =$$

$$= \frac{1}{4}\left(te^t - \int e^t dt\right) = \frac{1}{4}\left(te^t - e^t\right) = \frac{1}{4} e^t (t-1) =$$

$$= \frac{1}{4} e^{2x}(2x-1) + C.$$

2. $$\int x \sin 2x \, dx = \left| \begin{array}{l} t = 2x \\ dt = 2dx \\ \frac{dt}{2} = dx \end{array} \right| = \int \frac{t}{2} \sin t \frac{dt}{2} =$$

$$= \frac{1}{4} \int t \sin t \, dt = \left| \begin{array}{ll} f = t & f' = 1 \\ g' = \sin t & g = -\cos t \end{array} \right| =$$

$$= \frac{1}{4}\left(t(-\cos t) - \int(-\cos t) dt\right) = -\frac{1}{4}\left(t \cos t - \int \cos t \, dt\right)$$

$$= -\frac{1}{4}(t \cos t + \sin t) = -\frac{1}{4} \sin 2x - \frac{1}{2} x \cos 2x + c.$$

3. $$\int \cos\sqrt{x}\,dx = \left| \begin{array}{l} t=\sqrt{x} \\ dt=\frac{1}{2\sqrt{x}}\,dx \\ 2t\,dt=dx \end{array} \right| = \int \cos t \cdot 2t\,dt$$

$$= 2\int t\cos t\,dt = \left| \begin{array}{ll} f=t & f'=1 \\ g'=\cos t & g=\sin t \end{array} \right| =$$

$$= 2\left(t\sin t - \int \sin t\,dt\right) = 2(t\sin t + \cos t) =$$

$$= 2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + C$$

4. $$\int e^{\sqrt{x}}\,dx = \left| \begin{array}{l} t=\sqrt{x} \\ dt=\frac{1}{2}\frac{1}{\sqrt{x}}\,dx \\ 2t\,dt=dx \end{array} \right| = \int e^{t}\cdot 2t\,dt$$

$$= 2\int te^{t}\,dt = \left| \begin{array}{ll} f=t & f'=1 \\ g'=e^{t} & g=e^{t} \end{array} \right| =$$

$$= 2\left(te^{t} - \int e^{t}\,dt\right) = 2(te^{t} - e^{t}) = 2e^{\sqrt{x}}(\sqrt{x}-1)+C$$

5. $$\int \operatorname{arctg} 2x\,dx = \left| \begin{array}{l} t=2x \\ dt=2dx \\ dx=\frac{dt}{2} \end{array} \right| = \int \operatorname{arctg} t\cdot\frac{dt}{2}$$

$$= \frac{1}{2}\int \operatorname{arctg} t\,dt = \left| \begin{array}{ll} f=\operatorname{arctg} t & f'=\frac{1}{1+t^2} \\ g'=1 & g=t \end{array} \right|$$

$$= \frac{1}{2}\left(t\operatorname{arctg} t - \int\frac{t}{1+t^2}\,dt\right) = \frac{1}{2}t\operatorname{arctg} t - \frac{1}{4}\int\frac{2t}{1+t^2}\,dt$$

$$= \frac{1}{2}t\operatorname{arctg} t - \frac{1}{4}\ln(1+t^2) = x\operatorname{arctg} 2x - \frac{1}{4}(1+4x^2)+C$$

6. $\int \sin\sqrt{1+6x}\,dx = \left| \begin{matrix} t = 1+6x \\ dt = 6\,dx \\ dx = \frac{dt}{6} \end{matrix} \right| =$

$= \int \sin\sqrt{t}\cdot\frac{dt}{6} = \frac{1}{6}\int \sin\sqrt{t}\,dt =$

$= \left| \begin{matrix} u = \sqrt{t} & 2\sqrt{t}\,du = dt \\ & \Downarrow \\ du = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt & 2u\,du = dt \end{matrix} \right| =$

$= \frac{1}{6}\int \sin u \cdot 2u\,du = \frac{1}{3}\int u \sin u\,du$

$= \left| \begin{matrix} f = u & f' = 1 \\ g' = \sin u & g = -\cos u \end{matrix} \right| = \frac{1}{3}\left(-u\cos u + \int \cos u\,du\right)$

$= \frac{1}{3}(-u\cos u + \sin u) = \frac{1}{3}\left(-\sqrt{t}\cos\sqrt{t} + \sin\sqrt{t}\right) =$

$= \frac{1}{3}\left(-\sqrt{1+6x}\cos\sqrt{1+6x} + \sin\sqrt{1+6x}\right) + C$

Nebo stručněji:

$\left| \begin{matrix} u = \sqrt{1+6x} \\ du = \frac{1}{2}\frac{1}{\sqrt{1+6x}}\cdot 6\,dx \\ \frac{1}{3}\sqrt{1+6x}\,du = dx \\ \frac{1}{3}u\,du = dx \end{matrix} \right| \Rightarrow \int \sin u \cdot \frac{1}{3}u\,du$

$= \frac{1}{3}\int u \sin u\,du$

a dál je postup stejný!

Domácí cvičení: Kombinací metody substitucí a per partes vypočtěte integrály:

1. $\int \frac{\ln(\ln x)}{x}\,dx$, $V = \ln x \cdot \ln(\ln x) - \ln x + C$.

2. $\int \operatorname{arctg}\sqrt{x}\,dx$, $V = x \operatorname{arctg}\sqrt{x} + \operatorname{arctg}\sqrt{x} - \sqrt{x} + C$.

3. $\int \sin(\ln x)\,dx$, $V = \frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$.

Část IV. Integrace RLF a parciální zlomky

1. $$\int \frac{dx}{x(x^2+1)} = \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx \overset{*}{=}$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad / \cdot x(x^2+1)$$

$$1 = A(x^2+1) + (Bx+C)x \qquad A+B=0 \Rightarrow B=-1$$
$$1 = Ax^2 + A + Bx^2 + Cx \qquad C=0$$
$$1 = (A+B)x^2 + Cx + A \qquad A=1$$

$$\overset{*}{=} \int \frac{dx}{x} - \int \frac{x}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

Parciální zlomky a integrace rac. lom. fce

2. $\int \frac{3x-2}{x^2-6x+5}\,dx$

$D = 36 - 4\cdot 5 = 16 > 0 \Rightarrow$ rozklad na PZ

$x^2 - 6x + 5 = (x-1)(x-5)$

$$\frac{3x-2}{x^2-6x+5} = \frac{A}{x-1} + \frac{B}{x-5} \Rightarrow 3x-2 = A(x-5)+B(x-1)$$

a) $A + B = 3 \quad \Rightarrow \quad -4A = 1 \Rightarrow A = -\frac{1}{4}$

$-5A - B = -2 \qquad B = 3 - A = 3 + \frac{1}{4} = \frac{13}{4}$

$A = -\frac{1}{4},\ B = \frac{13}{4}$

Alternativa:

b) $x = 5 \Rightarrow 13 = 4B \Rightarrow B = \frac{13}{4}$

$x = 1 \Rightarrow 1 = -4A \Rightarrow A = -\frac{1}{4}$

Tedy

$$\int \frac{3x-2}{x^2-6x+5}\,dx = \int \frac{-\frac{1}{4}}{x-1} + \frac{\frac{13}{4}}{x-5}\,dx =$$

$$= -\frac{1}{4}\int \frac{dx}{x-1} + \frac{13}{4}\int \frac{dx}{x-5} =$$

$$= -\frac{1}{4}\ln|x-1| + \frac{13}{4}\ln|x-5| + C$$

Použili jsme vzorec $\int \frac{f'}{f} = \ln|f| + c$

3. $\int \frac{x-2}{x^2-7x+12}\,dx \quad = *$

$D = 49 - 4\cdot 12 = 1 > 0 \Rightarrow$ rozklad na PZ

$x^2 - 7x + 12 = (x-3)(x-4)$

$$\frac{x-2}{x^2-7x+12} = \frac{A}{x-3} + \frac{B}{x-4}$$

$x - 2 = A(x-4) + B(x-3)$

a) $A + B = 1 \quad \Rightarrow \quad B = 1 - 2A \qquad B = 2$

$-4A - 3B = -2 \qquad \Downarrow \; 4A + 3(1-A) = 2 \Rightarrow A = -1 \quad \Uparrow$

b) $x = 4 \Rightarrow B = 2$

$x = 3 \Rightarrow 1 = -A \Rightarrow A = -1$

$$\overset{*}{=} \int \frac{-1}{x-3} + \frac{2}{x-4}\,dx = \underline{-\ln|x-3| + 2\ln|x-4| + c}$$

4. $\int \frac{2x^2 + 41x - 91}{(x-1)(x+3)(x-4)}\,dx \quad = *$

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4} \quad \Big/ \cdot J$$

$2x^2 + 41x - 91 = A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x-3)$

Vhodná volba x

$x = -3: \quad 2\cdot 9 - 3\cdot 41 - 91 = B(-4)(-7) \Rightarrow B = -7$

$x = 4: \quad 2\cdot 16 + 4\cdot 41 - 91 = C\cdot 3\cdot 7 \Rightarrow C = 5$

$x = 1: \quad 2 + 41 - 91 = A\cdot 4\,(-3) \Rightarrow A = 4$

$$\overset{*}{=} \int \frac{4}{x-1} + \frac{-7}{x+3} + \frac{5}{x-4}\,dx = \underline{4\ln|x-1| - 7\ln|x+3| + 5\ln|x-4| + C}$$

5. $$\int \frac{x-1}{x^2+2x+5}\,dx$$

$$D = 4-4\cdot 5 = -16 < 0 \Rightarrow \text{P.Z}$$

$$\int \frac{x-1}{x^2+2x+5}\,dx = \frac{1}{2}\underbrace{\int \frac{2x+2}{x^2+2x+5}\,dx}_{A} - 2\underbrace{\int \frac{dx}{x^2+2x+5}}_{B} = *$$

$$A = \ln(x^2+2x+5)$$

$$B = \int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \begin{vmatrix} t = x+1 \\ dt = dx \end{vmatrix} =$$

$$= \int \frac{dt}{t^2+4} = \frac{1}{4}\int \frac{dt}{\left(\frac{t}{2}\right)^2+1} = \frac{1}{4}\int \frac{2du}{u^2+1} \quad \begin{vmatrix} \frac{t}{2} = u \\ \frac{1}{2}dt = du \end{vmatrix}$$

$$= \frac{1}{2}\operatorname{arctg} u = \frac{1}{2}\operatorname{arctg}\frac{t}{2} = \frac{1}{2}\operatorname{arctg}\frac{x+1}{2}$$

$$\overset{*}{=} \frac{1}{2}\ln(x^2+2x+5) - \operatorname{arctg}\frac{x+1}{2} + C.$$

6. $$\int \frac{x+2}{x^3-2x^2+2x}\,dx = *$$

$$\frac{x+2}{x^3-2x^2+2x} = \frac{x+2}{x(x^2-2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2-2x+2} \quad /\cdot J$$

$$x+2 = A(x^2-2x+2) + (Bx+C)x = (A+B)x^2 - $$
$$- (2A-C)x + 2A$$

soustava 3 rovnic o 3 neznámých

$$A+B = 0,\ -2A+C = 1,\ 2A = 2 \quad \Rightarrow$$

$$\Rightarrow A = 1,\ B = -1,\ C = 3$$

$$\overset{*}{=} \int \frac{dx}{x} - \int \frac{x-3}{x^2-2x+2}\,dx$$

$\quad\quad A \quad\quad\quad\quad B$

$A = \ln|x|$

$$B = \int \frac{x-3}{x^2-2x+2} = \frac{1}{2}\int \frac{2x-2}{x^2-2x+2} - 2\int \frac{dx}{x^2-2x+2} =$$

$$= \frac{1}{2}\ln(x^2-2x+2) - 2\int \frac{dx}{(x-1)^2+1} = \left| \begin{matrix} t = x-1 \\ dt = dx \end{matrix} \right|$$

$$= \frac{1}{2}\ln(x^2-2x+2) - 2\,\mathrm{arctg}(x-1)$$

Celkem:

$$\int \frac{x+2}{x^3-2x^2+2x} = \underline{\ln|x| - \frac{1}{2}\ln(x^2-2x+2) + 2\,\mathrm{arctg}(x-1) + C}$$

Domácí cvičení: RLF a parciální zlomky

1. $\int \frac{x^2}{x^2-1}\,dx$, $V = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$.

2. $\int \frac{x^2}{x^2-4x+3}\,dx$, $V = x + \frac{9}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + C$.

3. $\int \frac{2x^3+5x^2+8}{2x^2+7x-15}\,dx$, $V = \frac{x^2}{2} - x + 2\ln|2x-3| + 9\ln|x+5| + C$.

4. $\int \frac{x^3-2x^2+5x-4}{x^2-5x+6}\,dx$, $V = \frac{x^2}{2} + 3x + 20\ln|x-3| - 6\ln|x-2| + C$.

5. $\int \frac{x^2+3x+2}{x^2+x+2}\,dx$, $V = x + \ln(x^2+x+2) - \frac{2\sqrt{7}}{7}\mathrm{arctg}\frac{2x+1}{\sqrt{7}} + C$.

Příklady na další procvičení neurčitého integrálu: kombinace všech postupů.

1. $\int \frac{dx}{\sin x}$, V $= \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C.$

2. $\int \mathrm{tg}^3 x \, dx$, V $= \ln|\cos x| + \frac{1}{2\cos^2 x} + C.$

3. $\int \ln(2x+3)\,dx$, V $= x \ln|2x+3| + \frac{3}{2}\ln|2x+3| - x + C.$

URČITÝ INTEGRÁL

Část I: Substituční metoda

1. $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \, dx \\ 0 \mapsto \cos 0 = 1 \\ \frac{\pi}{2} \mapsto \cos\frac{\pi}{2} = 0 \end{array} \right| =$

$\int_1^0 t^2 \cdot (-dt) = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \underline{\underline{\frac{1}{3}}}$

2. $\int_0^1 \frac{x}{(x^2+1)^2} dx = \left| \begin{array}{l} t = x^2+1 \\ dt = 2x \, dx \\ 0 \mapsto 1 \\ 1 \mapsto 2 \end{array} \right| = \int_1^2 \frac{1}{t^2} \cdot \frac{dt}{2}$

$= \frac{1}{2} \int_1^2 t^{-2} dt = \frac{1}{2} \left[\frac{t^{-1}}{-1}\right]_1^2 = -\frac{1}{2}\left(\frac{1}{2} - 1\right) = \underline{\underline{\frac{1}{4}}}$

3. $\int_0^1 x\sqrt{1-x^2}\,dx = \left| \begin{array}{c} t = 1-x^2 \\ dt = -2x\,dx \\ 0 \mapsto 1 \\ 1 \mapsto 0 \end{array} \right| = \int_1^0 \sqrt{t}\cdot \frac{dt}{-2}$

$= \frac{1}{2}\int_0^1 \sqrt{t}\,dt = \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{1}{2}\left(\frac{1}{\frac{3}{2}} - 0\right) = \frac{1}{2}\cdot\frac{2}{3} = \underline{\underline{\frac{1}{3}}}$

Část II: Per-partes

1. $\int_0^{\frac{\pi}{2}} x\cos x\,dx = \left| \begin{array}{ll} f = x & f' = 1 \\ g' = \cos x & g = \sin x \end{array} \right| =$

$= \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1\cdot\sin x\,dx = \left(\frac{\pi}{2}\cdot 1 - 0\right) - \left[-\cos x\right]_0^{\frac{\pi}{2}} =$

$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{\pi}{2} - 1 \doteq 0{,}5708$

2. $\int_0^1 \operatorname{arctg} x\,dx = \left| \begin{array}{ll} f = \operatorname{arctg} x & f' = \frac{1}{1+x^2} \\ g' = 1 & g = x \end{array} \right|$

$= \left[x\operatorname{arctg} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \operatorname{arctg} 1 - \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2 = \underline{\underline{\frac{\pi}{4} - \ln\sqrt{2}}}$

3. $\int_0^{\pi} e^x \sin x\,dx = \left| \begin{array}{ll} f = e^x & f' = e^x \\ g' = \sin x & g = -\cos x \end{array} \right| =$

$= \left[-e^x\cos x\right]_0^{\pi} + \int_0^{\pi} e^x\cos x\,dx = \left| \begin{array}{ll} f = e^x & f' = e^x \\ g' = \cos x & g = \sin x \end{array} \right| =$

$$= \left[-e^x \cos x\right]_0^\pi + \left[e^x \sin x\right]_0^\pi - \int_0^\pi e^x \sin x \, dx$$

Celkem

$$2\int_0^\pi e^x \sin x \, dx = \left[-e^x \cos x\right]_0^\pi + \left[e^x \sin x\right]_0^\pi .$$

$$\int_0^\pi e^x \sin x \, dx = \frac{1}{2}\Big(-e^\pi \underbrace{\cos\pi}_{-1} + e^0 \underbrace{\cos 0}_{1} + e^\pi \underbrace{\sin\pi}_{0} - e^0 \underbrace{\sin 0}_{0}\Big)$$

$$= \underline{\underline{\frac{1}{2}\left(e^\pi + 1\right)}} \doteq 12{,}0703$$

Část III. Kombinace substituce a per-partes

1. $$\int_0^1 \arcsin x \, dx = \left| \begin{matrix} f = \arcsin x & f' = \frac{1}{\sqrt{1-x^2}} \\ g' = 1 & g = x \end{matrix} \right|$$

$$= \left[x \arcsin x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx = \left| \begin{matrix} t = \sqrt{1-x^2} \\ t^2 = 1-x^2 \\ 2t\,dt = -2x\,dx \\ 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix} \right| =$$

$$= \arcsin 1 - \int_1^0 \frac{-t\,dt}{t} =$$

$$= \frac{\pi}{2} - \int_0^1 dt = \frac{\pi}{2} - \left[t\right]_0^1 = \underline{\underline{\frac{\pi}{2} - 1}}$$

2. $$\int_0^{\sqrt{3}} x \operatorname{arctg} x \, dx = \left| \begin{matrix} f = \operatorname{arctg} x & f' = \frac{1}{1+x^2} \\ g' = x & g = \frac{x^2}{2} \end{matrix} \right|$$

$$= \left[\frac{x^2}{2} \operatorname{arctg} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x^2}{2(1+x^2)} \, dx = \frac{3}{2} \cdot \frac{\pi}{3} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{x^2+1} \, dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\sqrt{3}} 1 - \frac{1}{x^2+1} \, dx = \frac{\pi}{2} - \frac{1}{2}\left[x - \operatorname{arctg} x\right]_0^{\sqrt{3}} = \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

$$= \frac{2}{3}\pi - \frac{\sqrt{3}}{2} \doteq 1{,}2283$$

IV. Určitý integrál parc. zlomky a RLF

1. $$\int_0^{\frac{1}{2}} \frac{dx}{x^2-1} = \int_0^{\frac{1}{2}} \frac{dx}{(x-1)(x+1)} =$$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad / \cdot (x^2-1)$$

$$1 = A(x+1) + B(x-1)$$

$$x = -1 \implies 1 = -2B \implies B = -\frac{1}{2}$$

$$x = 1 \implies 1 = 2A \implies A = \frac{1}{2}$$

$$= \int_0^{\frac{1}{2}} \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\, dx = \frac{1}{2}\left(\int_0^{\frac{1}{2}} \frac{dx}{x-1} - \int_0^{\frac{1}{2}} \frac{dx}{x+1}\right)$$

$$= \frac{1}{2}\left(\Big[\ln|x-1|\Big]_0^{\frac{1}{2}} - \Big[\ln|x+1|\Big]_0^{\frac{1}{2}}\right) = \dots$$

$$= \underline{\underline{-\ln\sqrt{3}}} = -\frac{1}{2}\ln 3 \doteq 0{,}5493$$

2. $$\int_1^3 \frac{dx}{x^2+x} = \int_1^3 \frac{dx}{x(x+1)} = \int_1^3 \frac{1}{x} - \frac{1}{x+1}\, dx \overset{*}{=}$$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad / \,].$$

volím

$$1 = A(x+1) + Bx \qquad x = 0 \implies A = 1$$

$$x = -1 \implies B = -1$$

$$\overset{*}{=} \int_1^3 \frac{dx}{x} - \int_1^3 \frac{dx}{x+1} = \Big[\ln|x|\Big]_1^3 - \Big[\ln|x+1|\Big]_1^3 =$$

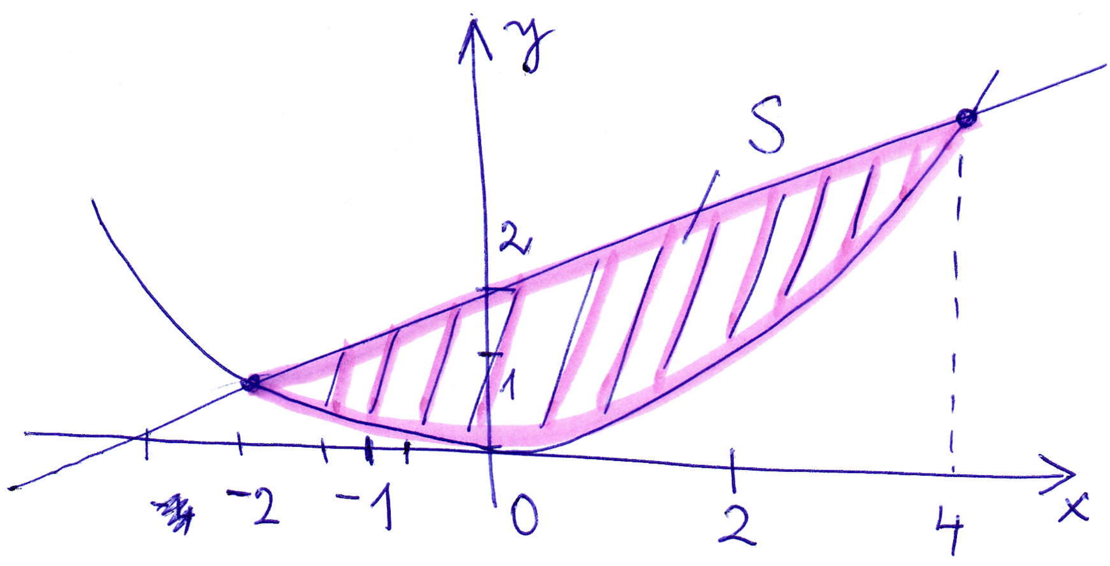
$$= \ln 3 - \ln 1 - (\ln 4 - \ln 2) = \ln\frac{3}{2} \doteq 0{,}4054$$

3. $\displaystyle\int_3^5 \frac{x-1}{x-2}\,dx = \int_3^5 1+\frac{1}{x-2}\,dx = \Big[x+\ln|x-2|\Big]_3^5$

$= 5+\ln 3 - 3 - \underbrace{\ln 1}_{=0} = \underline{\underline{2+\ln 3}}$

V. Aplikace určitého integrálu

1. Vypočtěte obsah části roviny omezené křivkami $y=\frac{x^2}{4}$, $y=\frac{x}{2}+2$.



$S=\displaystyle\int_{-2}^{4} \frac{x}{2}+2-\frac{x^2}{4}\,dx = \left[\frac{x^2}{4}+2x-\frac{x^3}{12}\right]_{-2}^{4} =$

$=\left(\frac{16}{4}+8-\frac{64}{12}\right)-\left(\frac{4}{4}-4+\frac{8}{12}\right) = \underline{\underline{9}}$

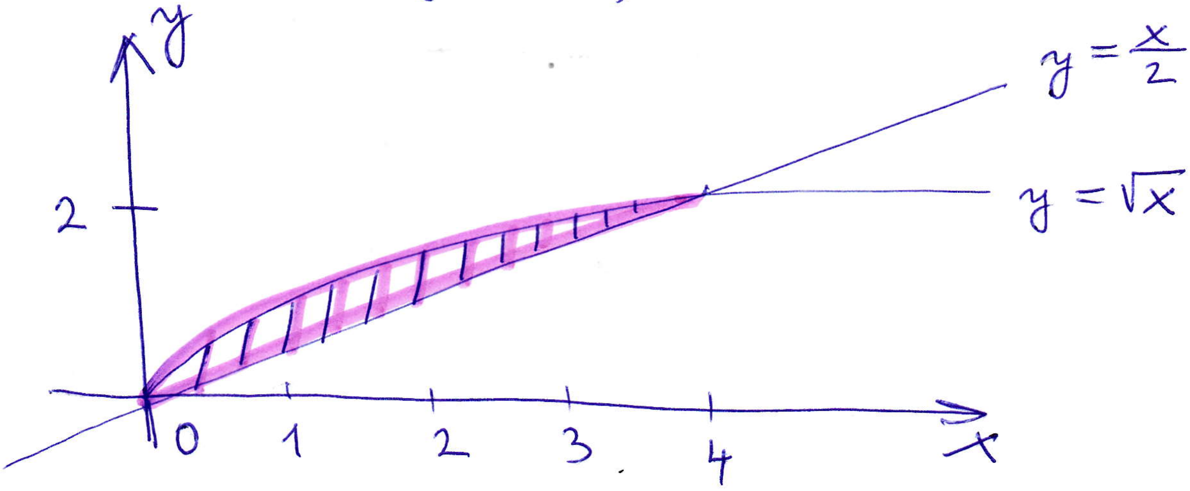
Meze integrálu jsme určili takto:

$\frac{x^2}{4} = \frac{x}{2}+2$

$x^2 = 2x+8$

$x^2-2x-8=0 \Leftrightarrow (x+2)(x-4)=0 \Leftrightarrow x=-2,\ x=4$

2. Vypočtěte obsah části roviny omezené křivkami $y=\sqrt{x}$, $y=\frac{x}{2}$.



$\sqrt{x}=\frac{x}{2} \Rightarrow x = \frac{x^2}{4} \Rightarrow 4x = x^2 \Rightarrow x^2-4x=0 \Rightarrow$
$x(x-4)=0 \Leftrightarrow x=0 \vee x=4$
Meze jsou 0 a 4.

$$S=\int_0^4\left(\sqrt{x}-\frac{x}{2}\right)dx = \int_0^4 x^{\frac{1}{2}}-\frac{x}{2}\,dx =$$
$$=\left[\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}\frac{x^2}{2}\right]_0^4 = \left[\frac{2}{3}\sqrt{x^3}-\frac{1}{4}x^2\right]_0^4 =$$
$$=\frac{2}{3}\cdot 8-\frac{1}{4}\cdot 16 = \frac{16}{3}-4 = \underline{\underline{\frac{4}{3}}}$$

3. Vypočtěte objem tělesa, které vznikne rotací křivky $y=\sqrt{\cos x\cdot e^{\sin x}}$ kolem osy x na intervalu $\langle 0, \frac{\pi}{2}\rangle$.

Vzorec: $V=\pi\int_a^b y^2(x)\,dx$.

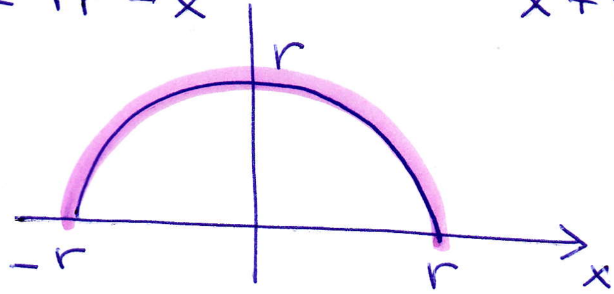
Dosadíme: $V=\pi\int_0^{\frac{\pi}{2}} \cos x\cdot e^{\sin x}\,dx =^{*}$

$$\overset{*}{=} \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \\ 0 \mapsto 0 \\ \frac{\pi}{2} \mapsto 1 \end{array} \right| = \pi \int_0^1 e^t \, dt = \pi \left[e^t \right]_0^1 = \underline{\underline{\pi (e-1)}}$$

4. Spočtěte velikost povrchu koule o poloměru r.

Vzorec: $P = 2\pi \int_a^b f(x) \cdot \sqrt{1+(f'(x))^2} \, dx$

$f(x) = \sqrt{r^2 - x^2}$



$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

$\oplus$

$$f'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2-x^2}}$$

$$P = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{1 + \left(\frac{1}{2} \cdot \frac{-2x}{\sqrt{r^2-x^2}} \right)^2} \, dx =$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} \, dx$$

$$= 2\pi \int_{-r}^{r} \sqrt{r^2} \, dx = 2\pi \int_{-r}^{r} r \, dx =$$

$$= 2\pi r \left[x \right]_{-r}^{r} = 2\pi r (r - (-r)) = \underline{\underline{4\pi r^2}}$$

5. Spočtěte délku závitu pravotočivé šroubovice o poloměru 1.

$x(t) = \cos t, \ y(t) = \sin t, \ z(t) = t, \ t \in \langle 0, 2\pi \rangle$

VZOREC:

$$l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

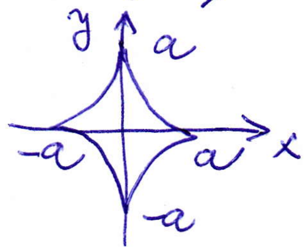
$x'(t) = -\sin t, \ y'(t) = \cos t, \ z'(t) = 1$

$$l = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt =$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\,[t]_0^{2\pi} = 2\sqrt{2}\,\pi \doteq 8{,}8857$$

6. Spočtěte délku asteroidy

$x(t) = a\cos^3 t, \ y(t) = a\sin^3 t, \ t \in \langle 0, 2\pi \rangle$



VZOREC:

$$l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$x'(t) = -3a\cos^2 t \cdot \sin t, \ y'(t) = 3a\sin^2 t \cdot \cos t$

$$l = 4\int_0^{\pi/2} \sqrt{(-3a\cos^2 t \sin t)^2 + (3a\sin^2 t \cos t)^2}\, dt$$

$$= 4\int_0^{\pi/2} 3a\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}\, dt =$$

$$= 4 \cdot 3a \int_0^{\pi/2} \cos t \sin t\, dt = \cancel{12a} \qquad \left| \begin{array}{l} u = \sin t \\ du = \cos t\, dt \end{array} \right| \quad \begin{array}{l} 0 \to 0 \\ \pi/2 \to 1 \end{array}$$

$$= 12a\int_0^1 u\, du = 12a\left[\frac{u^2}{2}\right]_0^1 = \underline{\underline{6a}}$$

Nevlastní integrály (i) vlivem meze

1. $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx =$$

$$\lim_{t\to\infty} \left[\ln x\right]_1^t = \lim_{t\to\infty} (\ln t - \underbrace{\ln 1}_{0}) =$$

$$\lim_{t\to\infty} \ln t = \underline{\underline{\infty}}$$ (integrál diverguje)

2. $$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx =$$

$$\lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} - \left(-\frac{1}{2}\right)\right) =$$

$$\lim_{t\to\infty} \left(\frac{1}{2} - \frac{1}{2t^2}\right) = \frac{1}{2} - \frac{1}{2\infty^2} = \frac{1}{2} - 0 = \underline{\underline{\frac{1}{2}}}$$

(integrál konverguje)

3. $$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx =$$

$$\lim_{t\to\infty} \left[\operatorname{arctg} x\right]_0^t = \lim_{t\to\infty} \operatorname{arctg} t = \underline{\underline{\frac{\pi}{2}}}$$

(integrál konverguje)

4. $$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t =$$

$$\lim_{t\to\infty} (-e^{-t} - (-e^{-1})) = \lim_{t\to\infty} \left(\frac{1}{e} - e^{-t}\right) =$$

$$= \frac{1}{e} - 0 = \underline{\underline{\frac{1}{e}}}$$

(ii) vlivem funkce

5.

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} =$$

$$= \lim_{t\to 1^-} \left[\arcsin x\right]_0^t = \lim_{t\to 1^-} \arcsin t - \underbrace{\arcsin 0}_{0}$$

$$= \lim_{t\to 1^-} \arcsin t = \underline{\underline{\frac{\pi}{2}}}$$

6.

$$\int_0^1 \frac{dx}{\sqrt[3]{x^2}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt[3]{x^2}} = *$$

$$\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-\frac{2}{3}}\,dx = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} = 3\sqrt[3]{x}$$

$$* = \lim_{t\to 0^+} \left[3\cdot\sqrt[3]{x}\right]_t^1 = \lim_{t\to 0^+} 3 - 3\cdot\sqrt[3]{t} = \underline{\underline{3}}$$

7.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx =$$

$$= \lim_{t\to 0^+} \left[x\ln x - x\right]_t^1 = \lim_{t\to 0^+} (1\cdot\underbrace{\ln 1}_{0} - 1) - (t\ln t - t)$$

$$= \lim_{t\to 0^+} t - t\ln t - 1 = \underbrace{\lim_{t\to 0^+} t}_{0} - \lim_{t\to 0^+} t\ln t - 1$$

$$= -1 - \lim_{t\to 0^+} t\ln t = |0\cdot\infty| = -1 + \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}$$

$$= -1 + \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = -1 + \lim_{t\to 0^+} (-t) = \underline{\underline{-1}}$$