Theory of Mathematical Structures

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To the memory of Charles Ehresmann

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Preface

Different branches of mathematics, e.g., algebra, topology or combinatorics, use different means to express their concepts and methods. It is thus surprising to see how much they actually have in common. The fundamentals of a great number of mathematical theories are built up on certain general principles. The study of these principles is the aim of the theory of mathematical structures and, more abstractly, the theory of categories.

Our book presents the theory of mathematical structures in a way comprehensible to a reader having but little experience with any concrete structure. We explain all the concepts used and exhibit a number of examples.

The concept of mathematical structures was introduced by N. Bourbaki in the *Theory of Sets* (Hermann, Paris, 1957). Chapter 4 "Structures" starts as follows:

The aim of the present chapter is to describe, once and for all, some of the constructions and proofs met particularly often in mathematics.

The description which Bourbaki used was, unfortunately, rather clumsy. Simultaneously, a more abstract (and more convenient) theory of categories was introduced by S. Eilenberg and S. MacLane. The theory they presented in their pioneering papers during the forties has been rapidly developed in the following decades. Today, there is a number of mathematicians working in the field of category theory, and still more those using categorical language in their work in other fields, ranging from topology and analysis to computer science.

In our book we present a nontraditional view of categories by returning somewhat to the concrete approach of Bourbaki. Our stress is on sets endowed with a structure and on mappings preserving this structure: such a setup is called a *construct*. We investigate the basic concepts concerning constructs: subobject, free object, initial structure, Cartesian product, etc. This is the contents of the first two chapters. Not until the third chapter do we introduce categories and functors, and we then study the interrelationship of various constructs (and categoris) and present some more abstract concepts. The fundamentals of the theory of categories are exhibited in the third and fourth chapters.

The last two chapters are devoted to a deeper theory of embedding of constructs and categories into special constructs: the algebraic and relational constructs and the construct of sets. The character of these two chapters is somewhat different from that of the preceding four. Most of the presented results appear for the first time in a book. The exposition is quicker and the demands on the reader are greater: e.g., we work here with ordinals and the transfinite induction.

Organization

Sections are denoted by capital letters, chapters and subsections by numbers. Thus,

5D3

designates chapter 5, section D, subsection 3.

The exercises of each section are denoted by case letters. Thus,

5Da

designates exercise a in section 5D. The exercises are usually easy and they are frequently referred to in the text.

A great number of names of constructs and categories is used (e.g., ***Top***, ***Gra***, etc.). A list of these names, as well as a list of other frequently used symbols, can be found at the end of the book. Also all historical comments are placed there.

Long proofs and arguments are concluded with the sign $\square$.

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PART I: Constructs

Chapter 1: Objects and Morphisms

1A. Sets

1. The purpose of this book is to study sets with a structure and structure-preserving maps. Without going into unnecessary technical details, we want to explain now what we mean by "sets".

We use the term "set" naively, i.e., we are not going to present a collection of axioms of a set theory, but we assume that the notion of a set is known to the reader. Each set X is determined by its elements, i.e., by the elements x such that $x \in X$. For example, the void set

$$\emptyset$$

has no elements; the set

$$\{x\}$$

has just one element x.

We are using the standard operations on sets: union, intersection, complement, Cartesian product, the power-set $\exp X$ (of all subsets of X), i.e.,

$$\exp X = \{M;\ M \subseteq X\},$$

and the set of maps,

$$Y^X = \{f;\ f \text{ is a map from } X \text{ to } Y\}.$$

When a set is written with the use of indices, e.g.,

$$X = \{x_i;\ i \in I\},$$

we call X a *collection* or *family* with the index set I.

Two sets X and Y are equal if each element of X is also an element of Y, and vice versa. Two collections $X = \{x_i;\ i \in I\}$ and $Y = \{y_j;\ j \in J\}$ are equal if $I = J$ and $x_i = y_i$ for each $i \in I$. Thus, the set $\{a_1, a_2\}$ is equal to the set $\{a_2, a_1\}$, but these two collections are different unless $a_1 = a_2$. For each family of sets $\{X_i;\ i \in I\}$ we can form the union $\bigcup_{i \in I} X_i$ and the intersection $\bigcap_{i \in I} X_i$.

Some standard symbols for sets are $\mathbb{N} = \{0, 1, 2, \ldots\}$, the natural numbers; $\mathbb{Z}$, the integers and $\mathbb{R}$, the real numbers.

2. Recall that a *map* is a triple*) consisting of a set X (the domain), a set Y (the range) and a relation $f \subseteq X \times Y$ such that for each $x \in X$ there exists a unique $y \in Y$ with $f(x) = y$ (i.e., with $(x, y) \in f$). We write this triple as

$$f\colon X \to Y.$$

For example, for each set X we have the *identity* map

$$\mathrm{id}_X\colon X \to X$$

defined by

$$\mathrm{id}_X(x) = x \qquad (x \in X).$$

If $X = \emptyset$ then the void relation $\emptyset \subseteq \emptyset \times Y$ is a map (because a statement "for each $x \in \emptyset \ldots$" is true by default); we call it the *void map*. Thus, for each set X we have

$$X^\emptyset = \{\text{void map}\}.$$

but

$$\emptyset^X = \emptyset \quad \text{if } X \neq \emptyset,$$
$$\emptyset^\emptyset = \{\mathrm{id}_\emptyset\} = \{\text{void map}\}.$$

Given maps

$$f\colon X \to Y \quad \text{and} \quad g\colon Y \to Z$$

the *composite map* is the map

$$g \cdot f\colon X \to Z$$

defined by

$$g \cdot f(x) = g(f(x)) \qquad \text{for each } x \in X.$$

3. Some families are "too large" to form a set. For example, we cannot form the "set of all sets". (This would lead to the famous Russel's paradox: denote by A the set of all sets X such that $X \notin X$. Then either $A \in A$, but this would imply $A \notin A$; or, $A \notin A$, but this would imply $A \in A$.) In the theory of mathematical structures we often work with such families, e.g., of all sets, of all vector spaces, etc. We need a broader concept than a set – we call it a *class*. Thus, classes are families generalizing sets in the following sense:

(1) each set is a class;

(2) for each property P of sets we can form the class $\{X;\ X \text{ is a set satisfying } P\}$.

For example, all sets form a class A; all sets X such that $X \notin X$ form a subclass A_0 of A. Neither A nor A_0 is a set (thus, $A_0 \notin A_0$ and this leads to no contradiction). A class which is not a set is called *large*. For contrast, sets are also called *small classes*.

We extend some of the set-theoretical operations to classes. Given classes X and Y, we form their Cartesian product, i.e., the class $X \times Y$ of all pairs (x, y) with

*) Pairs, triples, etc. are always assumed to be ordered.

$x \in X$ and $y \in Y$. Then we can define *class maps* $f: X \to Y$ quite similarly as above. We can also form the *union of classes* indexed by a class I. That is, given a class C_i for each $i \in I$, we can form the class

$$C = \bigcup_{i \in I} C_i,$$

the elements of which are precisely the elements of C_i for all $i \in I$. Finally, for each set X and each class Y we form the class Y^X of all class maps from X to Y. No other operations on classes will be used.

Classes are also written as collections

$$X = \{x_i;\ i \in I\}.$$

If I is a proper class we call X a *large collection*, reserving the simple term collection for the case of small index sets.

We use the *axiom of choice* for classes: if $\sim$ is an equivalence relation on a class X, then there exists a choice subclass, i.e., a class $Y \subseteq X$ such that each $x \in X$ is equivalent to precisely one $y \in Y$.

The reader acquainted with set theory will realize that we are working within an arbitrary theory of two universes, e.g., Bernays-Gödel theory or Zermelo-Fraenkel theory with a fixed universum, assuming the axiom of choice.

1B. Constructs: Definitions and Examples

1. Before presenting the definition of a construct, we illustrate some of its features on the case of (real) vector spaces.

In the theory of vector spaces there are two basic concepts: vector space and linear map. A vector space is a set X together with operations

$$+: X \times X \to X \quad \text{and} \quad \cdot: \mathbb{R} \times X \to X,$$

satisfying the well-known axioms. Formally, a vector space is a pair

$$(X, (+, \cdot))$$

consisting of a set X and its "structure" $(+, \cdot)$.

Let

$$(X, (+, \cdot)) \quad \text{and} \quad (Y, (+', \cdot'))$$

be two vector spaces. A map

$$f: X \to Y$$

is linear if it "preserves the structure", i.e., if

$$f(x_1 + x_2) = f(x_1) +' f(x_2) \qquad \text{for all} \quad x_1, x_2 \in X;$$
$$f(r \cdot x) = r \cdot' f(x) \qquad \text{for all} \quad x \in X;\ r \in \mathbb{R}.$$

If f is a linear map, we write $f: (X, (+, \cdot)) \to (Y, (+', \cdot'))$. Note the following properties of linear maps:

(i) The composition of linear maps is linear, i.e., if

$$f: (X, (+, \cdot)) \to (Y, (+', \cdot'))$$

and

$$g: (Y, (+', \cdot')) \to (Z, (+'', \cdot''))$$

are linear maps, then

$$g \cdot f: (X, (+, \cdot)) \to (Z, (+'', \cdot''))$$

is also a linear map.

(ii) For each vector space $(X, (+, \cdot))$ the identity map is linear:

$$\mathrm{id}_X: (X, (+, \cdot)) \to (X, (+, \cdot)).$$

These two properties of structure-preserving maps are encountered in numerous instances of "structures". Therefore, they serve as a basis for the following general definition.

2. Definition. A *construct* (or a concrete category of sets with structure) $\mathscr{S}$ is given by the following data:

a) For each set X a class $\mathscr{S}[X]$ is defined. Its elements are called the *structures* of X, and pairs

$$A = (X, \alpha),$$

where X is a set and α is its structure, are called *objects*.

b) For each pair of objects

$$A = (X, \alpha) \quad \text{and} \quad B = (Y, \beta)$$

a set

$$\hom_{\mathscr{S}}(A, B) \subseteq Y^X$$

is defined. Its elements are called the *morphisms* and, given a map $f: X \to Y$ then instead of $f \in \hom_{\mathscr{S}}(A, B)$, we write

$$f: A \to B.$$

The sets of morphisms satisfy the following axioms:

COMPOSITION AXIOM. The composition of two morphisms

$$f: A \to B \quad \text{and} \quad g: B \to C$$

is a morphism

$$g \cdot f: A \to C.$$

IDENTITY MAP AXIOM. For each object $A = (X, \alpha)$ the identity map is a morphism

$$\mathrm{id}_X: A \to A.$$

3. Example: the construct $\mathbf{Vect}$ is given as follows. For each set X,

$$\mathbf{Vect}[X]$$

denotes the set of all pairs $(+,\cdot)$ defining the structure of a vector space on X. Given objects, i.e., vector spaces, $A = (X,(+,\cdot))$ and $B = (Y,(+',\cdot'))$, then

$$\hom(A, B) \subseteq Y^X$$

is the set of all linear maps from A to B.

4. Terminology. Important constructs are denoted by an abbreviation of the names of their objects (e.g., $\mathbf{Vect}$). A list of these abbreviations can be found at the end of this book.

It is usual to state what the objects of a construct are rather than to introduce the classes $\mathscr{S}[X]$. For each object

$$A = (X, \alpha)$$

we call X the *underlying set* of A.

For each morphism

$$f\colon A \to B$$

we call A the *domain* and B the *range* of f. The identity morphism of an object $A = (X, \alpha)$ is often denoted by

$$1_A\colon A \to A$$

in place of $\mathrm{id}_X\colon A \to A$. We write $\hom(A, B)$ instead of $\hom_{\mathscr{S}}(A, B)$.

Remarks. (i) The class $\mathscr{S}[X]$ of structures of a set X can be empty. For example, if X is any finite set with at least two points then it does not carry the structure of any vector space. Thus,

$$\mathbf{Vect}[X] = \emptyset .$$

(ii) For technical reasons, the classes $\mathscr{S}[X]$ are usually supposed to be pairwise disjoint. In other words, each structure carries the information what underlying set is considered. We shall use this harmless convention.

5. Examples of constructs

(i) The construct $\mathbf{Pos}$ of posets (i.e., partially ordered sets) and order-preserving maps. Its objects, called *posets*, are pairs $(X, \leqq)$, where X is a set and $\leqq$ is an *ordering* on X, i.e., a binary relation which is

reflective ($x \leqq x$ for all $x \in X$),
antisymmetric ($x \leqq y$ and $y \leqq x$ imply $x = y$ for all $x, y \in X$) and
transitive ($x \leqq y$ and $y \leqq z$ imply $x \leqq z$ for all $x, y, z \in X$).

The morphisms from a poset $(X, \leqq)$ to a poset $(Y, \preceq)$ are the *order-preserving maps*, i.e, maps

$$f\colon X \to Y$$

such that

$$x_1 \leqq x_2 \quad \text{implies} \quad f(x_1) \preceq f(x_2)$$

for all $x_1, x_2 \in X$.

We must verify the axioms. First, let

$$f\colon (X, \leqq) \to (Y, \preceq) \quad \text{and} \quad g\colon (Y, \preceq) \to (Z, \sqsubseteq)$$

be order- preserving maps. Then

$$g \cdot f\colon (X, \leqq) \to (Z, \sqsubseteq)$$

is also order-preserving, since for all $x_1, x_2 \in X$,

$$x_1 \leqq x_2 \quad \text{implies} \quad f(x_1) \preceq f(x_2)$$

and

$$f(x_1) \preceq f(x_2) \quad \text{implies} \quad g(f(x_1)) \sqsubseteq g(f(x_2)).$$

Also,

$$\mathrm{id}_X\colon (X, \leqq) \to (X, \leqq)$$

is order-preserving since $x_1 \leqq x_2$ implies $x_1 \leqq x_2$ for all $x_1, x_2 \in X$.

The verification of the two axioms is usually quite routine and we leave it to the reader.

(ii) The construct $\mathbf{Gra}$ of graphs and compatible maps. Its objects, called *graphs*, are pairs (X, α), where X is a set and α is a binary relation, i.e.,

$$\alpha \subseteq X \times X .$$

In other words,

$$\mathbf{Gra}[X] = \exp(X \times X),$$

the set of all subsets of $X \times X$.

The morphisms from a graph (X, α) to a graph (Y, β) are the *compatible maps*, i.e., maps

$$f\colon X \to Y$$

such that

$$x_1 \alpha x_2 \quad \text{implies} \quad f(x_1) \beta f(x_2)$$

for all $x_1, x_2 \in X$

(iii) The construct $\mathbf{Set}$ of sets and maps. Its objects are (non-structured) sets and its morphisms are (all) maps. Formally, for each set X the class of structures $\mathbf{Set}[X]$ has just one element, say $*$; that is,

$$\mathbf{Set}[X] = \{*\} .$$

Thus an object $(X, *)$ can be identified with the set X. And for arbitrary objects X and Y,

$$\hom(X, Y) = Y^X.$$

Remark. The void set can, but need not, carry a structure. For example, each vector space has at least one element (the zero vector), thus,

$$\mathbf{Vect}[\emptyset] = \emptyset.$$

On the other hand, the void relation defines a graph, in fact a poset, on $\emptyset$. Thus, both ***Gra***$[\emptyset]$ and ***Pos***$[\emptyset]$ are singleton classes.

6. **Definition.** A *subconstruct* of a construct $\mathcal{S}$ is a construct $\mathcal{T}$ such that

a) each object of $\mathcal{T}$ is an object of $\mathcal{S}$, i.e.,

$$\mathcal{T}[X] \subseteq \mathcal{S}[X] \quad \text{for each set } X$$

and

b) each morphism of $\mathcal{T}$ is a morphism of $\mathcal{S}$, i.e.,

$$\hom_{\mathcal{T}}(A, B) \subseteq \hom_{\mathcal{S}}(A, B)$$

for arbitrary objects A and B of $\mathcal{T}$.

And $\mathcal{T}$ is a *full subconstruct* if

$$\hom_{\mathcal{T}}(A, B) = \hom_{\mathcal{S}}(A, B)$$

for arbitrary objects A and B of $\mathcal{T}$.

For example, ***Pos*** is a full subconstruct of ***Gra***. Each poset is a graph, and given posets $A = (X, \leq)$ and $B = (Y, \preceq)$ then a map $f: X \to Y$ is order-preserving iff it is compatible. Thus,

$$\hom_{\mathbf{Pos}}(A, B) = \hom_{\mathbf{Gra}}(A, B).$$

7. **Example:** The construct ***Lat*** of lattices and lattice homomorphisms.

A *lattice* is a poset $(X, \leq)$ in which each pair $x_1, x_2 \in X$ has a join $x_1 \vee x_2$ (i.e. the least of all elements $y \in X$ satisfying $x_1 \leq y$ and $x_2 \leq y$) and a meet $x_1 \wedge x_2$ (i.e., the largest of all elements $z \in X$ satisfying $x_1 \geq z$ and $x_2 \geq z$). For example, on the set $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ we can define the *ordering by division*:

$$x \preceq y \quad \text{iff } x \text{ divides } y \quad (x, y \in \mathbb{Z}^+).$$

Then $(\mathbb{Z}^+, \preceq)$ is a lattice: $x \vee y$ is the least common multiple of x and y, while $x \wedge y$ is their greatest common divisor $(x, y \in \mathbb{Z}^+)$. Also the usual ordering $\leq$ of $\mathbb{Z}^+$ defines a lattice $(\mathbb{Z}^+, \leq)$: here $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

On the other hand, the *discrete order* on a set X, which is defined by $x_1 \leq x_2$ iff $x_1 = x_2$, does not yield a lattice (unless X has at most one element).

lattice = svaz

The objects of the construct ***Lat*** are all lattices. The morphisms from a lattice $(X, \leq)$ to a lattice $(Y, \preceq)$, called *lattice homomorphisms*, are maps $f: X \to Y$ preserving joins and meets, i.e., such that

$$\begin{aligned} f(x_1 \vee x_2) &= f(x_1) \vee f(x_2) \\ f(x_1 \wedge x_2) &= f(x_1) \wedge f(x_2) \end{aligned} \qquad \text{for all} \quad x_1, x_2 \in X.$$

Observations. (i) ***Lat*** is a subconstruct of ***Pos***. Indeed, each lattice is a poset. Also, given lattices $(X, \leq)$ and $(Y, \preceq)$, then each lattice homomorphism

$$f: (X, \leq) \to (Y, \preceq)$$

is order-preserving: if $x_1, x_2 \in X$ fulfil

$$x_1 \leq x_2$$

then obviously $x_1 \vee x_2 = x_2$. This implies

$$f(x_1) \leq f(x_1) \vee f(x_2) = f(x_1 \vee x_2) = f(x_2).$$

(ii) ***Lat*** is not a full subconstruct of ***Pos***. In other words, there exists an order-preserving map $f: (X, \leq) \to (Y, \preceq)$ which is not a homomorphism, though both $(X, \leq)$ and $(Y, \preceq)$ are lattices.

For example, consider the lattices $(\mathbb{Z}^+, \preceq)$ and $(\mathbb{Z}^+, \leq)$ defined above. The identity map is clearly order-preserving:

$$\mathrm{id}_Z : (\mathbb{Z}^+, \preceq) \to (\mathbb{Z}^+, \leq).$$

But it is not a lattice homomorphism since it fails to preserve the meet $2 \wedge 3 = 1$.

8. **Example:** The construct ***Clat*** of complete lattices and complete lattice homomorphisms.

A poset $(X, \leq)$ is a *complete lattice* if each subset $T \subseteq X$ has a join $\bigvee T$ (i.e., the least of all elements $y \in X$ satisfying $t \leq y$ for each $t \in T$) and a meet $\bigwedge T$ (i.e., the largest of all elements $z \in X$ satisfying $t \geq z$ for each $t \in T$). For example, the set

$$X = \exp M$$

of all subsets of a set M ordered by the inclusion $\subseteq$ is a complete lattice. For each $T \subseteq X$ (i.e., for each collection T of subsets of M) the union of all sets in T is the join of T:

$$\bigvee T = \bigcup_{M' \in T} M'$$

and the intersection of all sets in T is the meet:

$$\bigwedge T = \bigcap_{M' \in T} M'.$$

On the other hand, the lattice $(\mathbb{Z}^+, \preceq)$, defined in 1B7, is not complete: it does not have, for example, the join of $T = \mathbb{Z}^+$ (i.e., it has no largest element).

Note that in each complete lattice the join of $\emptyset$ is, by definition, the least element (because any $y \in X$ satisfies $t \leqq y$ for each $t \in \emptyset$, by default). Analogously with the meet of $\emptyset$: $\bigwedge \emptyset = \bigvee X$ and $\bigvee \emptyset = \bigwedge X$.

The construct ***Clat*** has as objects all complete lattices. Its morphisms from $(X, \leqq)$ to $(Y, \leqq)$, called *complete lattice homomorphisms*, are maps

$$f\colon X \to Y$$

which preserve the join and the meet of each subset $T \subseteq X$. Thus, denoting

$$f(T) = \{f(t);\ t \in X\},$$

we have

$$\bigvee f(T) = f(\bigvee T)$$

and

$$\bigwedge f(T) = f(\bigwedge T).$$

Observation. ***Clat*** is a subconstruct of ***Lat*** which is not full. The affirmative part is evident; thus, it suffices to find a lattice homomorphism between complete lattices which is not a complete homomorphism.

Consider the usual extension of real numbers $(\mathbb{R}^+, \leqq)$, where

$$\mathbb{R}^+ = \mathbb{R} \cup \{+\infty, -\infty\}.$$

The map

$$f\colon (\mathbb{R}^+, \leqq) \to (\mathbb{R}^+, \leqq)$$

defined by

$$f(x) = 0 \quad (x \in \mathbb{R});$$

$$f(-\infty) = -\infty \text{ and } f(+\infty) = +\infty$$

is a lattice homomorphism. But it is not complete

$$\bigvee \mathbb{R} = +\infty,$$

but

$$\bigvee f(\mathbb{R}) = 0.$$

9. Remark. A poset $(X, \leqq)$, in which each subset $T \subseteq X$ has a meet $\bigwedge T$, can be called a *complete semilattice* (more in detail, a complete meet-semilattice). This term is, however, unneeded since each complete semilattice is a complete lattice.

Let T be a subset of a complete semilattice $(X, \leqq)$. The join of T is, by definition, the least element of the following set

$$T^+ = \{y \in X;\ t \leqq y \text{ for each } t \in T\}.$$

Therefore, $\bigvee T$ exists because $\bigwedge T^+$ exists, and we have

$$\bigvee T = \bigwedge T^+.$$

Nevertheless, the term complete semilattice is useful with respect to morphisms! Given complete (semi-)lattices $(X, \leqq)$ and $(Y, \leqq)$ then a map

$$f\colon X \to Y$$

is called a *complete semilattice homomorphism* if it preserves all meets, i.e., if

$$f(\bigwedge T) = \bigwedge f(T) \qquad \text{for each} \quad T \subseteq X.$$

Let ***Csl*** denote the construct of complete semilattices and complete semilattice homomorphisms. Then

Csl is a subconstruct of ***Clat***

and these two constructs have the same objects. But ***Csl*** is not a full subconstruct (in other words, ***Csl*** $\neq$ ***Clat***). Choose any set M with at least two points and choose $m_0 \in M$; define

$$f\colon (\exp M, \subseteq) \to (\exp M, \subseteq)$$

as follows:

$$f(M') = \begin{cases} M' & \text{if } m_0 \in M' \\ \emptyset & \text{else} \end{cases} \qquad \text{for each } M' \subseteq M.$$

Then f is a complete semilattice homomorphisms, i.e.,

$$f(\bigcap_{i \in I} M_i) = \bigcap_{i \in I} f(M_i)$$

for each collection $\{M_i;\ i \in I\}$ of subsets of M. On the other hand, f is not even a lattice homomorphism:

$$f(\{m_0\}) \vee f(M - \{m_0\}) = \{m_0\};$$

$$f(\{m_0\} \vee (M - \{m_0\})) = M.$$

Concluding remark. The choice of objects of a construct does not determine the choice of morphisms: two (naturally arising) constructs with the same objects need not have the same morphisms. We have seen several examples of non-full subconstructs:

$$\textit{Clat} \subseteq \textit{Csl} \subseteq \textit{Lat} \subseteq \textit{Pos}.$$

That these subconstructs are not full is caused by the fact that their morphisms are required to preserve less and less structure (from the left to the right). On the other hand, ***Pos*** is a full subconstruct of ***Gra***, since in these two constructs morphisms are defined by the same "rule". Finally

Pos and ***Vect***

are incompatible constructs: neither is a subconstruct of the other one.

Exercises 1B

a. The constructs $\mathbf{Lat}_p$ of partial lattices: its objects are all posets; its morphisms are all maps $f\colon (X, \leq) \to (Y, \leq)$ which preserve all the existing joins and meets of pairs, i.e.,

$$x_1 \vee x_2 = x \quad \text{implies} \quad f(x_1) \vee f(x_2) = f(x)$$

for all $x_1, x_2 \in X$ for which $x_1 \vee x_2$ exists; analogously with $x_1 \wedge x_2$.

(1) What are the interrelations of the constructs ***Pos***, $\mathbf{Lat}_p$ and ***Lat***?

(2) Let $(X, \leq)$ be a discrete poset (i.e., $x_1 \leq x_2$ implies $x_1 = x_2$); prove that for each poset $(Y, \leq)$ and each map $f\colon X \to Y$ we have a morphism $f\colon (X, \leq) \to (Y, \leq)$ of $\mathbf{Lat}_p$.

b. Preordered sets are pairs $(X, \leq)$ where $\leq$ is a reflexive and transitive (but not necessarily antisymmetric) relation. The construct ***Pros*** of preordered sets is defined as the full subconstruct of ***Gra***, the objects of which are all preordered sets.

(1) Check that the "ordering by the norm" in $\mathbb{R}^n$:

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \quad \text{iff} \quad \sqrt{\sum a_i^2} \leq \sqrt{\sum b_i^2}$$

(for $(a_i), (b_i) \in \mathbb{R}^n$) is a preorder.

(2) Check that each equivalence relation is a preorder.

(3) For each preordered set $A = (X, \leq)$ verify that the relation

$$x_1 \sim x_2 \quad \text{iff both} \quad x_1 \leq x_2 \text{ and } x_2 \leq x_1 \quad (x_1, x_2 \in X)$$

is an equivalence relation on X.

(4) In (3), let $X/\sim$ be the set of all equivalence classes $[x] = \{t \in X;\ t \sim x\}$ for $x \in X$. Verify that $x \leq y$ implies $t \leq s$ for all $t \in [x]$ and $s \in [y]$. Define a relation on $X/\sim$ as follows:

$$[x] \leq^* [y] \quad \text{iff } x \leq y \text{ (for each } [x], [y] \in X/\sim).$$

Verify that $A^* = (X/\sim, \leq^*)$ is a poset. Terminology: A^* is called the *antisymmetrization* of A.

(5) Find the antisymmetrization of the preordered sets in (1), (2).

c. Normed vector spaces. Recall that a norm on a vector space $(X, +, \cdot)$ is a map $\|\ \|\colon X \to [0, +\infty]$ such that

(i) $\|x\| = 0$ iff $x = 0$ (the zero vector), for each $x \in X$;

(ii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in X$;

(iii) $\|r \cdot x\| = |r| \cdot \|x\|$ for each $x \in X, r \in \mathbb{R}$.

Denote by ***Nor*** the construct of normed vector spaces, i.e., quadruples $(X, +, \cdot, \|\ \|)$ consisting of a vector space $(X, +, \cdot)$ and its norm $\|\ \|$, and the norm-decreasing linear maps. Thus, the morphisms from $(X, +, \cdot, \|\ \|)$ into $(Y, +', \cdot', \|\ \|')$ are those linear maps $f\colon (X, +, \cdot) \to (X', +', \cdot')$ which fulfil the following condition:

$$\|f(x)\|' \leq \|x\| \quad \text{for each } x \in X.$$

(1) Verify that the Euclidean space $(\mathbb{R}^n, +, \cdot)$ with its usual norm $\|a\| = \sqrt{\sum a_i^2}$ for each $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, is a normed vector space.

(2) For which $k \in \mathbb{R}$ is the linear map $f(x) = k \cdot x$ a morphism $f\colon (\mathbb{R}^n, +, \cdot, \|\ \|) \to (\mathbb{R}^n, +, \cdot, \|\ \|)$?

(3) Denote by $\mathrm{Mat}_{2,2}$ the set of all (2, 2)-matrices. Verify that it is a normed vector space under the usual addition and scalar multiplication of matrices and with the following norm

$$\|A\| = |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}| \quad \text{for each } A = \begin{bmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{bmatrix} \in \mathrm{Mat}_{2,2}.$$

Let $K = (k_{ij})$ be a matrix with $|k_{ij}| = \frac{1}{2}$; prove that the map $f\colon \mathrm{Mat}_{2,2} \to \mathrm{Mat}_{2,2}$, defined by $f(X) = K \cdot X$, is a morphism.

1C. Isomorphisms

1. When studying the objects of a certain construct, it is important to know when two of them are to be considered "esentially" the same. For example, two vector spaces are "esentially" the same iff they have the same dimension. The exact formulation of these considerations is expressed by the notion of isomorphism.

Recall that a map $f\colon X \to Y$ is a *bijection* if it is one-to-one and onto. Equivalently, if there exists a map $f^{-1}\colon Y \to X$ with

$$f^{-1} . f = \mathrm{id}_X \quad \text{and} \quad f . f^{-1} = \mathrm{id}_Y .$$

Then f^{-1} is called the *inverse* of f.

* 2. Definition. An *isomorphism* is a morphism

$$f\colon (X, \alpha) \to (Y, \beta)$$

such that f is a bijection and the inverse map is a morphism

$$f^{-1}\colon (Y, \beta) \to (X, \alpha).$$

Remark. In this definition we did not state explicitly what construct is considered. More precisely, the definition should be as follows: Let $\mathcal{S}$ be a construct, then its morphism $f\colon (X, \alpha) \to (Y, \beta)$ is an isomorphism if .... Whenever an arbitrary (but fixed) construct is considered, we leave its symbol out of definitions or theorems.

Terminology. Two objects (X, α) and (Y, β) are called *isomorphic* if there exists an isomorphism $f\colon (X, \alpha) \to (Y, \beta)$; in symbols

$$(X, \alpha) \cong (Y, \beta).$$

Note that a bijective morphism need not be an isomorphism. For example, the morphism $\mathrm{id}_{\mathbb{Z}^+}: (\mathbb{Z}^+, \leqq) \to (\mathbb{Z}^+, \lesssim)$ (in ***Pos***) of Observation 1B7(ii) is bijective. But this morphism is not an isomorphism, since the inverse map is not order-preserving $\mathrm{id}_{\mathbb{Z}^+}: (\mathbb{Z}^+, \lesssim) \longrightarrow (\mathbb{Z}^+, \leqq)$.

Examples. (i) In the construct ***Pos*** of posets, isomorphisms are morphisms $f: (X, \leqq) \to (Y, \lesssim)$ such that f is a bijection "transporting" the relation $\leqq$ onto the relation $\lesssim$ in the sense that

$$x_1 \leqq x_2 \quad \text{iff} \quad f(x_1) \lesssim f(x_2) \quad \text{for each} \quad x_1, x_2 \in X .$$

Thus, two posets $(X, \leqq)$ and $(Y, \lesssim)$ are isomorphic iff one is obtained by a "relabelling" of the elements of the other. For example, the posets

$$(\exp\{1, 2, 3\}, \subseteq) \quad \text{and} \quad (\exp\{a, b, c\}, \subseteq)$$

are isomorphic.

(ii) Two vector spaces are isomorphic (in the construct ***Vect***) iff they have the same dimension.

(iii) Two finite sets are isomorphic (in the construct ***Set***) iff they have the same number of elements.

3. Remark. The relation $\cong$, to be isomorphic, is an equivalence relation on the class of all objects (of any construct). Indeed:

(i) For each object (X, α),

$$\mathrm{id}_X: (X, \alpha) \to (X, \alpha)$$

is an isomorphism (because $\mathrm{id}_X = \mathrm{id}_X^{-1}$); thus,

$$(X, \alpha) \cong (X, \alpha) .$$

(ii) For each isomorphism

$$f: (X, \alpha) \to (Y, \beta)$$

the inverse map is also an isomorphism

$$f^{-1}: (Y, \beta) \to (X, \alpha)$$

(because $(f^{-1})^{-1} = f$); thus

$$(X, \alpha) \cong (Y, \beta) \quad \text{implies} \quad (Y, \beta) \cong (X, \alpha) .$$

(iii) The composition of two isomorphisms

$$f: (X, \alpha) \to (Y, \beta) \quad \text{and} \quad g: (Y, \beta) \to (Z, \gamma)$$

is an isomorphism

$$g . f: (X, \alpha) \to (Z, \gamma)$$

(because $(g . f)^{-1} = f^{-1} . g^{-1}$); thus

$$(X, \alpha) \cong (Y, \beta) \quad \text{and} \quad (Y, \beta) \cong (Z, \gamma) \quad \text{imply} \quad (X, \alpha) \cong (Z, \gamma) .$$

4. Example: the construct ***Met*** of metric spaces and contractions. Recall that a *metric* on a set X is a map $\alpha: X \times X \to [0, +\infty)$ such that for all $x_1, x_2, x_3 \in X$,

(i) $\alpha(x_1, x_2) = 0$ iff $x_1 = x_2$;
(ii) $\alpha(x_1, x_2) = \alpha(x_2, x_1)$;
(iii) $\alpha(x_1, x_2) + \alpha(x_2, x_3) \geqq \alpha(x_1, x_3)$.

The construct ***Met*** has as objects all *metric spaces*, i.e., pairs (X, α) where X is a set and α is its metric. The morphisms from a space (X, α) to a space (Y, β), called *contractions*, are maps

$$f: X \to Y$$

such that

$$\beta(f(x_1), f(x_2)) \leqq \alpha(x_1, x_2) \qquad \text{for all} \quad x_1, x_2 \in X .$$

An example of a metric space is the n-dimensional *Euclidean space* (of n-tuples of real numbers)

$$(\mathbb{R}^n, \varrho)$$

where given $x_1 = (a_1, \dots, a_n)$ and $x_2 = (b_1, \dots, b_n)$ in $\mathbb{R}^n$, we put

$$\varrho(x_1, x_2) = \sqrt{\sum (a_i - b_i)^2} .$$

(In particular, in $\mathbb{R} = \mathbb{R}^1$ we have $\varrho(x_1, x_2) = |x_1 - x_2|$.)

Remark. Isomorphisms in ***Met*** are called *isometries*. An isometry from a space (X, α) to a space (Y, β) is a bijection

$$f: X \to Y$$

which "transports" the metric α onto the metric β in the sense that

$$\beta(f(x_1), f(x_2)) = \alpha(x_1, x_2) \qquad \text{for all} \quad x_1, x_2 \in X .$$

For example, the linear map $f(x) = kx + q$ is an isometry of $(\mathbb{R}, \varrho)$ iff $|k| = 1$. Thus, the intervals

$$[0, 1] \quad \text{and} \quad [5, 6]$$

in $\mathbb{R}$ with the Euclidean metric $\varrho(x_1, x_2) = |x_1 - x_2|$ are isometric, i.e., isomorphic in ***Met***. Indeed, the map

$$f: [0, 1] \to [5, 6] ,$$

defined by $f(x) = x + 5$ $(x \in [0, 1])$, is an isometry.

5. Example: the construct ***Top*** of topological spaces and continuous maps. Recall that a *topology* on a set X is a collection α of its subsets (called *open sets*) such that

(i) $\emptyset$ and X are open, i.e., $\{\emptyset, X\} \subseteq \alpha$;
(ii) the intersection of two open sets is open, i.e.,

$$M_1, M_2 \in \alpha \quad \text{implies} \quad M_1 \cap M_2 \in \alpha;$$

(iii) the union of open sets is open, i.e.,

$$M_i \in \alpha \quad \text{for each} \quad i \in I \quad \text{implies} \quad \bigcup_{i \in I} M_i \in \alpha.$$

The construct ***Top*** has as objects *topological spaces*, i.e., pairs (X, α) where X is a set and α is its topology. The morphisms from a space (X, α) to a space (Y, β) are all *continuous maps*, i.e., maps

$$f\colon X \to Y$$

such that the preimage of each open set is open:

$$M \in \beta \quad \text{implies} \quad f^{-1}(M) \in \alpha \qquad (M \subseteq Y).$$

Each metric α on X *induces* a topology $\tilde{\alpha}$: a set $M \subseteq X$ is defined to be open iff for each $m \in M$ there exists a number $r \in (0, +\infty)$ such that

$$\alpha(x, m) < r \quad \text{implies} \quad x \in M \qquad (\text{for all } x \in X).$$

If there is no danger of confusion, we use the same symbol for a metric and the induced topology.

For example, the line $(\mathbb{R}, \varrho)$ is a topological space in which a set $M \subseteq \mathbb{R}$ is open iff with each point $m \in M$ it contains an open interval with the midpoint m. A set M is open in the plane $(\mathbb{R}^2, \varrho)$ iff with each point $m \in M$ it contains a disc with the centre m.

Each contraction is continuous. More precisely, for each morphism in ***Met***

$$f\colon (X, \alpha) \to (Y, \beta)$$

we have a morphism in ***Top***

$$f\colon (X, \tilde{\alpha}) \to (Y, \tilde{\beta}).$$

Let $M \subseteq Y$ be open. To prove that $f^{-1}(M)$ is open, choose any point $m \in f^{-1}(M)$. Since $f(m) \in M$ and M is open, there exists $r \in (0, +\infty)$ such that

$$\beta(y, f(m)) < r \quad \text{implies} \quad y \in M.$$

Then also

$$\alpha(x, m) < r \quad \text{implies} \quad x \in f^{-1}(M),$$

because if $\alpha(x, m) < r$ then $\beta(f(x), f(m)) < r$ (since f is a contraction); hence, $f(x) \in M$.

On the other hand, a continuous map need not be a contraction. For example, each linear map $f(x) = kx + q$ is continuous

$$f\colon (\mathbb{R}, \varrho) \to (\mathbb{R}, \varrho).$$

In fact, continuous maps from $(\mathbb{R}, \varrho)$ into itself are precisely the continuous functions as defined in the calculus.

Remark. Isomorphisms in ***Top*** are called *homeomorphisms*. A homeomorphism from a space (X, α) to a space (Y, β) is a bijection

$$f\colon X \to Y$$

which "transports" the topology α onto the topology β in the sense that

$$M \in \alpha \quad \text{iff} \quad f(M) \in \beta \qquad \text{for each} \quad M \subseteq X.$$

For example, all closed intervals $[a, b]$ in $\mathbb{R}$, with the topology ϱ induced by the Euclidean metric, are pairwise homeomorphic (i.e., isomorphic in ***Top***). Indeed, given two closed intervals $[a, b]$ and $[a', b']$ there clearly exists a linear map

$$f(x) = kx + q, \qquad k \neq 0,$$

mapping $[a, b]$ onto $[a', b']$. Then

$$f\colon ([a, b], \varrho) \to ([a', b'], \varrho)$$

is a homeomorphism, because the inverse map is also linear (hence, continuous):

$$f^{-1}(y) = \frac{y - q}{k} \qquad (\text{for each } y \in [a', b']).$$

Note, for example, that

$$([0, 1], \tilde{\varrho}) \cong ([0, 2], \tilde{\varrho}) \qquad \text{in } \textbf{\textit{Top}}$$

but

$$([0, 1], \varrho) \not\cong ([0, 2], \varrho) \qquad \text{in } \textbf{\textit{Met}}.$$

Indeed, we have

$$\varrho(0, 2) = 2$$

and no two points in $[0, 1]$ have distance 2; thus, $[0, 1]$ and $[0, 2]$ are not isometric.

6. Example: the construct ***Topm*** of metrizable spaces and continuous maps. A topological space (X, α) is said to be *metrizable* if there exists a metric γ on X, inducing the given topology, i.e., such that

$$\alpha = \tilde{\gamma}.$$

We denote by ***Topm*** the full subconstruct of ***Top***, the objects of which are all metrizable topological spaces (and morphisms are, necessarily, all continuous maps between metrizable spaces – this follows from the fullness).

An example of a non-metrizable topological space is the *indiscrete space* (X, α) where $\alpha = \{\emptyset, X\}$. If X has at least two points, then for each metric γ we have $\{\emptyset, X\} \subsetneqq \tilde{\gamma}$: choose $x_1, x_2 \in X$ with $x_1 \neq x_2$; then $\gamma(x_1, x_2) > 0$ and the set

$$M = \{x \in X;\ \gamma(x, x_2) < \gamma(x_1, x_2)\}$$

is evidently open, $M \in \tilde{\gamma}$. Yet, $M \neq \emptyset$, because $x_2 \in M$, and $M \neq X$, because $x_1 \notin M$.

An example of a metrizable space is the *discrete space* (X, α), in which each set is open:

$$\alpha = \exp X\,.$$

This topology is induced, e.g., by the following metric:

$$\gamma(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases} \qquad (x_1, x_2 \in X)\,.$$

(Each set M is open in $\tilde{\gamma}$ because for every $m \in M$ we can choose $r = 1$; then $\alpha(x, m) < r$ implies $x = m \in M$.)

Distinct metrics can induce the same topology. E.g., if γ is a metric inducing the discrete topology, then the metric 2γ also induces it. Thus, the objects of the constructs ***Met*** and ***Topm*** are basically different.

7. We have seen in the above examples that isomorphisms are just those bijections which "transport" one structure onto another. It is an important property of most of the usual constructs that bijections can transport structures in the following sense.

Definition. A construct $\mathscr{S}$ is said to be *transportable* if for each object (X, α) and each bijection $f\colon X \to Y$ there exists precisely one structure $\beta \in \mathscr{S}[Y]$ such that

$$f\colon (X, \alpha) \to (Y, \beta)$$

is an isomorphism.

Examples. (i) ***Gra*** is transportable. The unique β is defined by

$$\beta = \{(y_1, y_2) \in Y \times Y;\quad \text{there exists } (x_1, x_2) \in \alpha \text{ with } y_1 = f(x_1) \text{ and } y_2 = f(x_2)\}\,.$$

Moreover, if α is an ordering, then so is β – hence, ***Pos*** is also transportable. If α is a lattice ordering then so is β – hence, ***Lat*** is transportable.

(ii) ***Met*** is transportable. The unique β is defined by

$$\beta(y_1, y_2) = \alpha(x_1, x_2) \quad \text{where} \quad f(x_1) = y_1 \quad \text{and} \quad f(x_2) = y_2\,.$$

This follows from Remark 1C4.

(iii) ***Top*** is transportable. The unique β is defined by

$$M \in \beta \quad \text{iff} \quad f^{-1}(M) \in \alpha\,.$$

This follows from Remark 1C5.

Let us mention an example of a construct which fails to be transportable.

8. Example: the construct ***Mtc*** of metric spaces and continuous maps. Recall that a map $f\colon X \to Y$ is *continuous* from a metric space (X, α) to a metric space (Y, β) if for each $x \in X$ and each $\varepsilon \in (0, +\infty)$ there exists a $\delta \in (0, +\infty)$ such that

$$\alpha(x, t) < \delta \quad \text{implies} \quad \beta(f(x), f(t)) < \varepsilon \qquad \text{for each} \quad t \in X\,.$$

This is equivalent to the continuity of f from $(X, \tilde{\alpha})$ to $(Y, \tilde{\beta})$.

The construct ***Mtc*** of metric spaces and continuous maps is not transportable: if two metrics α_1, α_2 on a set X induce the same topology, i.e., if

$$\tilde{\alpha}_1 = \tilde{\alpha}_2\,,$$

then

$$\mathrm{id}_X\colon (X, \alpha_1) \to (X, \alpha_2)$$

is an isomorphism of ***Mtc***, of course. Assume $\alpha_1 \neq \alpha_2$ (for example, $\alpha_1 = 2\alpha_2$); then

$$\mathrm{id}_X\colon (X, \alpha_1) \to (X, \alpha_1) \quad \text{and} \quad \mathrm{id}_X\colon (X, \alpha_1) \to (X, \alpha_2)$$

are isomorphisms, in contradiction to the uniqueness of β in the definition above.

9. Concluding remark. For each construct, the class of all objects is partitioned into subclasses of pairwise isomorphic objects, i.e., objects which are (up to a relabelling of elements) equal. Isomorphisms are bijections which "transport" the structure. In most of the usual constructs, bijections conversely "transport" structure uniquely, thus determining the isomorphisms.

Exercises 1C

a. Isomorphisms of lattices. Prove that each isomorphism $f\colon (X, \leqq) \to (Y, \leqq)$ in ***Pos***, where $(X, \leqq)$ and $(Y, \leqq)$ are lattices, is already a lattice isomorphism (i.e., an isomorphism in ***Lat***). Does the same hold for complete lattices?

b. The line is homeomorphic to intervals. Prove that $(\mathbb{R}, \varrho)$ is homeomorphic to each open interval (a, b), the topology of which is the restriction of ϱ. Hint: the continuous map $f(x) = \tan x$ maps $(-\pi/2, \pi/2)$ onto $\mathbb{R}$; its inverse $f^{-1}(x) = \arctan x$ is also continuous. Thus, $(\mathbb{R}, \varrho) \cong ((-\pi/2, \pi/2), \varrho)$.

c. Closure in a topological space. Let (X, α) be a topological space. The *closure* of a subset $M \subseteq X$ is the set $\overline{M}$ of all points $x \in X$ such that

$$x \in U \quad \text{implies} \quad M \cap U \neq \emptyset \qquad (\text{for each } U \in \alpha)\,.$$

If $\overline{M} = M$ then M is said to be *closed*.

(1) What is the closure of $(0, 1)$, $(0, 1]$ and $[0, 1]$ in the line $(\mathbb{R}, \varrho)$?

(2) Prove that a set M is closed iff $X - M$ is open.

(3) Prove that the closure operation is
isotone: $M_1 \subseteq M_2$ implies $\overline{M}_1 \subseteq \overline{M}_2$ $(M_1, M_2 \in \exp X)$;
idempotent: $\overline{\overline{M}} = \overline{M}$ $(M \in \exp X)$;
additive: $\overline{\emptyset} = \emptyset$ and $\overline{M_1 \cup M_2} = \overline{M}_1 \cup \overline{M}_2$ $(M_1, M_2 \in \exp X)$.

(4) Given topological spaces (X, α) and (Y, β), prove that a map $f\colon X \to Y$ is continuous iff it respects the closure in the sense that

$$f(\overline{M}^{\alpha}) \subseteq \overline{f(M)}^{\beta} \qquad \text{for each} \quad M \subseteq X\,.$$

(5) Prove that a map is continous iff the preimage of each closed set is closed.

(6) Characterize $\overline{M}$ in a discrete space and in an indiscrete space.

d. Constructs of topological spaces. Topological spaces can be classified by properties related to the possibility to "separate" points and sets. For each of these properties we obtain a full subconstruct of ***Top***, the objects of which are all spaces with this property (and morphisms are all continuous maps).

(1) $\boldsymbol{Top}_0$, the construct of T_0-spaces. A topological space (X, α) is T_0 if each pair of distinct points $x, y \in X$ can be separated by an open set, i.e., if there is $U \in \alpha$ such that $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.

Is the indiscrete space a T_0-space?

Consider the following space, which is called the *Sierpinski space*: $X = \{a_1, a_2\}$ and the open sets are $\emptyset$, X and $\{a_2\}$. Is it T_0?

(2) $\boldsymbol{Top}_1$, the construct of T_1-spaces. A topological space (X, α) is T_1 if each finite subset is closed.

Prove that a space is T_1 iff each pair of distinct points $x, y \in X$ can be separated by open sets U, V in the sense that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Prove that the Sierpinski space is not T_1.

Define the *space of finite complements* on each infinite set X as follows: a set M is open iff $X - M$ is a finite set, or $M = \emptyset$. Prove that this is a T_1-space.

(3) $\boldsymbol{Top}_2$, the construct of T_2-spaces (or *Hausdorff* spaces). A topological space is T_2 if each pair of distinct points $x, y \in X$ can be separated by disjoint open sets, i.e., there exist disjoint open sets U, V with $x \in U$ and $y \in V$.

Prove that a space of finite complements is not T_2.

Prove that each metrizable space is T_2.

e. Compact spaces. Let (X, α) be a topological space, and let $M \subseteq X$ be a subset. An *open cover* of M is a collection U_i, $i \in I$, of open sets such that $M \subseteq \bigcup_{i \in I} U_i$. The set M is *compact* if for each of its open covers U_i, $i \in I$, there exists a finite open subcover, i.e., a finite set $J \subseteq I$ such that $M \subseteq \bigcup_{i \in J} U_i$. The space is compact if $M = X$ is a compact set.

(1) Prove that each indiscrete space is compact, while a discrete space is compact iff it is finite.

(2) Prove that the line $(\mathbb{R}, \varrho)$ is not compact. A non-trivial proposition: a set $M \subseteq \mathbb{R}$ is compact iff it is bounded (i.e., $M \subseteq [a, b]$ for some $a, b \in \mathbb{R}$) and closed.

(3) Let (X, α) be a compact T_2-space. Prove that a subset of X is compact iff it is closed. Hint: if M is closed and U_i, $i \in I$ is its open cover, then $X - M$ and U_i, $i \in I$, form an open cover of X. Conversely, if M is not closed, pick some $x \in \overline{M} - M$ and for each $m \in M$ choose disjoint open sets U_m, V_m with $m \in U_m$, $x \in V_m$. Then U_m, $m \in M$, is an open cover of M with no finite subcover.

(4) Prove that for each continuous map $f\colon (X, \alpha) \to (Y, \beta)$ and each compact set $M \subseteq X$ the set $f(M) \subseteq Y$ is also compact.
$M \subseteq \mathbb{R}$ is compact iff it is bounded (i.e., $M \subseteq [a, b]$ for some $a, b \in \mathbb{R}$) and closed.

f. Dense subsets. A subset of a topological space is *dense* if its closure is all of the space.

(1) Prove that a set M is dense iff it meets each nonvoid open set.

(2) Prove that the set of all rationals forms a dense set in the line $(\mathbb{R}, \varrho)$.

(3) Which subsets are dense in a discrete space?, in an indiscrete space?, in a space of finite complements?

(4) Prove that a subset M of a metric space (X, α) is dense (in the topology $\tilde{\alpha}$) iff for each $x \in X$

$$\bigwedge_{m \in M} \alpha(x, m) = 0\,.$$

1D. Fibres

1. Definition. Let $\alpha, \beta \in \mathscr{S}[X]$ be two structures on the same set X. We say that α is *finer* than β (or that β is *coarser* than α) if

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \beta)$$

is a morphism. We write

$$\alpha \leqq \beta$$

or, more precisely, $\alpha \leqq_{\mathscr{S}} \beta$.

Examples. (i) ***Gra***: given relations α, β on a set X, then

$$\alpha \leqq \beta \quad \text{iff} \quad \alpha \subseteq \beta\,.$$

Thus, a relation α is finer than β iff it contains fewer pairs. In particular, the finest relation is the void one, and the coarsest relation is all of $X \times X$.

Since ***Pos*** is a full subconstruct of ***Gra***, the analogous statement holds for posets. As an example, consider the orderings $\lesssim$ and $\leqq$ of the set $\mathbb{Z}^+$ (Example 1B7). In ***Pos***

$$\lesssim \text{ is finer than } \leqq\,.$$

However, in ***Lat***

$$\precsim \text{ is not finer than } \leqq .$$

(ii) ***Top***: given topologies α, β on a set X then $\alpha \leqq \beta$ iff each α-open set is β-open, i.e., iff $\beta \subseteq \alpha$ (as subsets of $\exp X$).

The finest topology is the discrete one, and the coarsest is the indiscrete one. Another example: consider the line $\mathbb{R}$ with the Euclidean topology ϱ; each finite set is closed (i.e., the line is T_1), hence, its complement is open – thus, ϱ is finer than the topology of finite complements (1Cd(2)).

2. Definition. A transportable construct $\mathscr{S}$ is said to be *fibre-small* if for each set X the class $\mathscr{S}[X]$ (of all structures on X) is small. In other words, if for each set X the collection of all objects with underlying set X is a set.

Proposition. Let $\mathscr{S}$ be a fibre-small construct. For each set X the relation "to be finer" defines a poset

$$(\mathscr{S}[X], \leqq),$$

called the *fibre* of the set X in the construct $\mathscr{S}$.

Proof. The relation $\leqq$ is

(1) reflexive since for each $\alpha \in \mathscr{S}[X]$,

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \alpha)$$

is a morphism;

(2) transitive since given α, β, γ in $\mathscr{S}[X]$ such that

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \beta) \quad \text{and} \quad \mathrm{id}_X\colon (X, \beta) \to (X, \gamma)$$

are morphisms, then so is

$$\mathrm{id}_X = \mathrm{id}_X \cdot \mathrm{id}_X\colon (X, \alpha) \to (X, \gamma);$$

(3) antireflexive since given α, β in $\mathscr{S}[X]$ such that U symetr

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \beta) \quad \text{and} \quad \mathrm{id}_X\colon (X, \beta) \to (X, \alpha)$$

are morphisms, then

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \beta)$$

is an isomorphism. Since

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \alpha)$$

is also isomorphism, the definition of transportability (1C7) implies

$$\alpha = \beta . \qquad \square$$

Remarks. (i) In all the constructs which we have introduced above, the classes $\mathscr{S}[X]$ are small. This is further true in all the constructs usually met in mathematics. Thus, for all practical purposes "fibre-small" and "transportable" are equivalent (and extremely mild) conditions on a construct.

(ii) A fibre-small construct is called *fibre-discrete* if all its fibres are discretely ordered, i.e., if for arbitrary $\alpha, \beta \in \mathscr{S}[X]$

$$\alpha \neq \beta \quad \text{implies} \quad \mathrm{id}_X\colon (X, \alpha) \to (X, \beta) \quad \text{is not a morphism} .$$

A fibre-small construct is called *fibre-complete* if all its fibres are complete lattices.

Examples. (i) ***Gra*** is a fibre-complete construct. The fibre of a set X is the set of all subsets of $X \times X$ ordered by inclusion (see the example (i) above),

$$(\boldsymbol{Gra}[X], \leqq) = (\exp X \times X, \subseteq) .$$

(ii) ***Top*** is a fibre-complete construct. For each collection of topologies

$$T \subseteq \boldsymbol{Top}[X]$$

define a topology α as follows:

$$\alpha = \bigcap T = \{M \subseteq X;\ M \text{ is open in each topology } \beta \in T\} .$$

(This is easily seen to be a topology.) Then

$$\alpha = \bigvee T \quad \text{in} \quad \boldsymbol{Top}[X] .$$

(iii) ***Pos*** is not fibre-complete: for each set X with at least two elements there exists no coarsest ordering on X. On the other hand, ***Pos*** is also not fibre-discrete (see the example (i) above).

(iv) ***Lat*** is fibre-discrete. If

$$\mathrm{id}_X\colon (X, \leqq) \to (X, \precsim)$$

is a lattice homomorphism then the meets (as well as joins) in the posets $(X, \leqq)$ and $(X, \precsim)$ coincide:

$$\begin{aligned} x_1 \wedge_{(\leqq)} x_2 &= \mathrm{id}_X(x_1 \wedge_{(\leqq)} x_2) \\ &= \mathrm{id}_X(x_1) \wedge_{(\precsim)} \mathrm{id}_X(x_2) = x_1 \wedge_{(\precsim)} x_2 . \end{aligned}$$

Hence, for all $x_1, x_2 \in X$,

$$x_1 \leqq x_2 \quad \text{iff} \quad x_1 = x_1 \wedge x_2 \quad \text{iff} \quad x_1 \precsim x_2 .$$

In other words, the orderings $\leqq$ and $\precsim$ are equal.

3. Algebraic structures. A number of important fibre-discrete constructs are the constructs of algebras. We review some of them briefly.

An *operation of arity n* on a set X is a map from the n-fold Cartesian product X^n (of all n-tuples in X) into X,

$$\alpha\colon X^n \to X .$$

In particular, we have the following:
a unary operation $\alpha\colon X \to X$;
a binary operation $\alpha\colon X \times X \to X$
(we usually write $x \cdot y$ or $x + y$, etc., instead of $\alpha(x, y)$);
a ternary operation $\alpha\colon X \times X \times X \to X$.
We also consider $n = 0$, the nullary operations.

Convention. For each set X put

$$X^0 = \{0\};$$

we often write $1 = \{0\}$; thus,

$$X^0 = 1.$$

A nullary operation is a map $\alpha\colon \{0\} \to X$, which is usually identified with the element $\alpha(0)$ of X.

A set endowed with a collection of operations is called an *algebra*. A map

$$f\colon X \to Y$$

is said to *preserve operations* α (on X) and β (on Y) of the same arity n, if for each n-tuple (x_i) in X^n

$$\alpha(x_i) = x \quad \text{implies} \quad \beta(f(x_i)) = f(x).$$

Maps preserving all the given operations are the morphisms in various algebraic constructs – they are usually called *homomorphisms*.

4. Examples. (i) A *groupoid* is an algebra $(X, \circ)$, where $\circ$ is a binary operation.*) The construct of groupoids and homomorphisms is denoted by ***Grd***; a homomorphism from a groupoid $(X, \circ)$ into a groupoid $(Y, \cdot)$ is a map $f\colon X \to Y$ such that

$$x_1 \circ x_2 = x \quad \text{implies} \quad f(x_1) \cdot f(x_2) = f(x) \qquad (x_1, x_2, x \in X).$$

As an example, consider the set $\mathbb{R}$ of real numbers with the addition $+$ and the multiplication $\cdot$. The exponential function $f(x) = e^x$ is a homomorphism

$$f\colon (\mathbb{R}, +) \to (\mathbb{R}, \cdot).$$

(ii) A *semigroup* is a groupoid $(X, \circ)$ satisfying the *associativity law*:

$$x_1 \circ (x_2 \circ x_3) = (x_1 \circ x_2) \circ x_3 \quad (x_1, x_2, x_2 \in X).$$

The construct ***Sgr*** of semigroups is a full subconstruct of ***Grd***.

*) The term groupoid appears also in a different context: as a construct (or category) in which each morphism is an isomorphism. We do not use this concept in our book.

Both $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ are semigroups. But $\mathbb{R}^+ = (0, +\infty)$ with the exponential operation $x_1 * x_2 = x_1^{x_2}$ is not a semigroup since for example $(2*2)*3 \neq 2*(2*3)$.

(iii) A *monoid* is an algebra $(X, \circ, e)$, where $(X, \circ)$ is a semigroup and e is its *unit element*, i.e., an element such that

$$e \circ x = x \quad \text{and} \quad x \circ e = x \qquad (x \in X);$$

e is considered as a nullary operation. The construct of monoids and homomorphisms is denoted by ***Mon***. A monoid homomorphism from $(X, \circ, e)$ to $(Y, \cdot, \hat{e})$ is a semigroup homomorphism f preserving the unit,

$$f(e) = \hat{e}.$$

For example, $(\mathbb{R}, +, 0)$ and $(\mathbb{R}, \cdot, 1)$ are monoids. And $f(x) = e^x$ is a monoid homomorphism.

Another example: for each set Σ (called an *alphabet*) denote by Σ^* the set of all *words*, or finite sequences, over Σ. The elements are $\emptyset$, the void word; σ_1, the one-letter words (for each $\sigma_1 \in \Sigma$); $\sigma_1\sigma_2$, the two-letter words (for each $\sigma_1, \sigma_2 \in \Sigma$) etc. Then $(\Sigma^*, \cdot, \emptyset)$ is a monoid, where $\cdot$ is the *concatenation* of words:

$$\sigma_1\sigma_2 \ldots \sigma_n \cdot \tau_1\tau_2 \ldots \tau_m = \sigma_1\sigma_2 \ldots \sigma_n\tau_1\tau_2 \ldots \tau_m$$

(for each $\sigma_1\sigma_2 \ldots \sigma_n$ and $\tau_1\tau_2 \ldots \tau_m$ in Σ^*).

(iv) A monoid $(X, \circ, e)$ is a *group* if for each element $x \in X$ there exists an *inverse element* x^{-1}, i.e., an element such that

$$x \circ x^{-1} = e \quad \text{and} \quad x^{-1} \circ x = e.$$

Denote by ***Grp*** the full subconstruct of ***Mon***, the objects of which are all groups.

Note that each homomorphism in ***Grp***

$$f\colon (X, \circ, e) \to (Y, \cdot, \hat{e})$$

preserves the inverse elements:

$$f(x^{-1}) = f(x)^{-1} \qquad \text{for each} \quad x \in X.$$

In fact, inverse elements are easily seen to be unique (if $x \circ y = e$ then $y = x^{-1}$ because $y = e \circ y = x^{-1} \circ x \circ y = x^{-1} \circ e = x^{-1}$) and we have

$$f(x^{-1}) \cdot f(x) = f(x^{-1} \circ x) = f(e) = \hat{e} \qquad (x \in X).$$

Therefore, we could consider groups as algebras

$$(X, \circ, e, in)$$

where in is the unary operation of inverse element. This would result in a formally different construct which is, however, "essentially" the same. We make these considerations precise in the next section.

$(\mathbb{R}, +, 0)$ is a group; $(\mathbb{R}, \cdot, 1)$ is not a group, since 0 has no inverse element; $(\mathbb{R} - \{0\}, \cdot, 1)$ is a group.

(v) The construct *Vect* of vector spaces can be naturally viewed as an algebraic construct: each vector space on a set X is given by the binary operation $+$ and by a collection of unary operations

$$r \cdot (\ ): X \to X \qquad (r \in \mathbb{R}),$$

assigning to each vector $x \in X$ its scalar multiple $r \cdot x$.

5. Observation. All the algebraic constructs in the preceding examples have discrete fibres. More generally, given n-ary operations α, β on a set X which are preserved by id_X, then

$$\alpha(x_i) = x \quad \text{implies} \quad \beta(x_i) = x \qquad \text{for each} \quad (x_i) \in X^n;$$

in other words, $\alpha = \beta$.

6. A different situation occurs with the fibres in constructs of partial algebras. A *partial algebra* is a set X together with partial operations, i.e., maps from subsets of X^n into X. The definition of a homomorphism of partial algebras has several natural variants — we present one of them, not going into general considerations (and restricting ourselves to a few examples only).

A *partial groupoid* is a pair $(X, \circ)$, where X is a set and $\circ$ is a map from a subset of $X \times X$ into X; thus, $x_1 \circ x_2$ is either an element of X or is undefined (for all $x_1, x_2 \in X$). A *homomorphism* from a partial groupoid $(X, \circ)$ into a partial groupoid $(Y, \cdot)$ is a map $f\colon X \to Y$ such that

$$x_1 \circ x_2 = x \quad \text{implies} \quad f(x_1) \cdot f(x_2) = f(x) \qquad (x_1, x_2 \in X).$$

Thus, whenever $x_1 \circ x_2$ is defined, so is $f(x_1) \cdot f(x_2)$. Denote by $\boldsymbol{Grd}_{\mathrm{p}}$ the construct of partial groupoids and homomorphisms.

Observation. The fibres in $\boldsymbol{Grd}_{\mathrm{p}}$ are not discrete. Given partial operations $\circ$ and $\cdot$ on a set X then $\circ$ is finer than $\cdot$ iff

$$x_1 \circ x_2 = x_1 \cdot x_2 \quad \text{whenever} \quad x_1 \circ x_2 \quad \text{is defined} \qquad (\text{for all } x_1, x_2 \in X).$$

Thus, the finest operation is that which is nowhere defined.

7. We conclude this section by a notion needed in the third chapter: a partial monoid. (This rather special concept is introduced for the purposes seen below. The term partial monoid is not currently used in algebra; thus our terminology does not contradict any current usage.)

A partial groupoid $(X, \circ)$ is said to be *weakly associative* if for arbitrary $x_1, x_2, x_3 \in X$ such that $x_1 \circ x_2$ and $x_2 \circ x_3$ are defined, we have

$$(x_1 \circ x_2) \circ x_3 = x_1 \circ (x_2 \circ x_3).$$

The equality is understood in the sense that both sides are defined.

A *unit* in a partial groupoid $(X, \circ)$ is an element $e \in X$ such that for each $x \in X$,

$$\text{if} \quad x \circ e \quad \text{is defined then} \quad x \circ e = x;$$
$$\text{if} \quad e \circ x \quad \text{is defined then} \quad e \circ x = x.$$

A *partial monoid* is a triple $(X, \circ, E)$, where $(X, \circ)$ is a weakly associative partial groupoid, $E \subseteq X$ is a set of units and for each $x \in X$ there exist units $e_x \in E$ and ${}_xe \in E$ such that

$$x \circ e_x = x \quad \text{and} \quad {}_xe \circ x = x.$$

A *homomorphism* from a partial monoid $(X, \circ, E)$ into a partial monoid $(Y, \cdot, \hat{E})$ is a map $f\colon X \to Y$ which is a homomorphism of the partial groupoids preserving units, i.e.,

$$e \in E \quad \text{implies} \quad f(e) \in \hat{E}.$$

The resulting construct is denoted by $\boldsymbol{Mon}_{\mathrm{p}}$.

Example. Let X be the set of all (real) matrices, let $\cdot$ be the usual multiplication of matrices, and let E be the set of all the unit matrices. Then $(X, \cdot, E)$ is a partial monoid: for each (n, k)-matrix $x \in X$ the unit e_x is the unit (k, k)-matrix and ${}_xe$ is the unit (n, n)-matrix.

Observation. The fibres in $\boldsymbol{Mon}_{\mathrm{p}}$ are not discrete. For example, on the set X of all real matrices define an operation $\circ$ which is the usual multiplication $x \circ y$ if x or y is a unit matrix, and which is undefined otherwise. Then $(\circ, E)$ is finer than $(\cdot, E)$.

Exercises 1D

a. Transformation monoids. A map from a set X into itself (i.e., an element of X^X) is called a *transformation*. A transformation monoid is a set $T \subseteq X^X$ containing id_X and closed under composition ($f, g \in T$ implies $f \,.\, g \in T$).

(1) Verify that $(T, ., \mathrm{id}_X)$ is a monoid.

(2) Prove that each monoid $(X, \circ, e)$ is isomorphic to a transformation monoid. Hint: for each $x \in X$ define a transformation $t_x \in X^X$ by $t_x(y) = x \circ y$ $(y \in X)$. Then $t_x \,.\, t_{x'} = t_{x \cdot x'}$ and $t_e = \mathrm{id}_X$.

b. Abelian groups are groups $(X, +, 0)$ satisfying the *commutativity law*,

$$x_1 + x_2 = x_2 + x_1 \qquad (x_1, x_2 \in X).$$

The construct of Abelian groups is denoted by $\boldsymbol{Ab}$; it is a full subconstruct of $\boldsymbol{Grp}$. (The operation of an Abelian group is usually denoted by $+$ and its unit by 0.)

(1) Verify that the addition of integers defines an Abelian group $(\mathbb{Z}, +, 0)$. What about the multiplication of integers?

$(X, \{p_i : i \in I\})$ [illegible] [illegible] $\tau = (n_i : i \in I)$

[illegible] $id_X : (X, \{p_i : i \in I\}) \mapsto (X, \{q_i : i \in I\})$ [illegible] [illegible]

$p_i(x_1, \dots, x_{n_i}) = id_X(p_i(x_1, \dots, x_{n_i})) = q_i(id_X(x_1), \dots, id_X(x_{n_i})) = q_i(x_1, \dots, x_{n_i})$

(2) Two integers $x, y \in Z$ are said to be *congruent modulo* $k = 1, 2, 3, \ldots$ if k divides $|x - y|$; in symbols, $x \equiv y \pmod{k}$. Put

$$\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$$

and define the addition on Z_k as follows: $x \oplus y$ is the unique element of $\mathbb{Z}_k$, congruent to the usual sum $x + y$ modulo k. Analogously with multiplication $\odot$.

Prove that $(\mathbb{Z}_k, \oplus, 0)$ is an Abelian group.

Prove that $(\mathbb{Z}_k - \{0\}, \odot, 1)$ is an Abelian group iff k is a prime.

(3) Find a non-Abelian group. Hint: the largest transformation monoid defining a group.

c. Rings and fields. A (unitary) *ring* is an algebra $(X, +, 0, \cdot, 1)$, where $(X, +, 0)$ is an Abelian group and $(X, \cdot, 1)$ is a monoid such that the following *distributive laws* hold:

$$\begin{aligned} x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ (y + z) \cdot x &= (y \cdot x) + (z \cdot x) \end{aligned} \qquad \text{for all} \quad x, y, z \in X .$$

A *field* is a ring such that $(X - \{0\}, \cdot, 1)$ is a group. The construct of rings is denoted by ***Rng***; its morphisms are the ring homomorphisms which are monoid homomorphisms with respect to both of the operations $+$ and $\cdot$. The full subconstruct of fields is denoted by ***Fld***.

(1) Verify that $(\mathbb{R}, +, 0, \cdot, 1)$ is a field.

(2) Verify that $(\mathbb{Z}_k, \oplus, 0, \odot, 1)$ is a ring which is a field iff k is a prime.

(3) Verify that the map

$$f\colon (\mathbb{Z}, +, 0, \cdot, 1) \to (\mathbb{Z}_k, \oplus, 0, \odot, 1)$$

which assigns to each integer $z \in \mathbb{Z}$ the remainder of the integer division $|z|: k$ is a ring homomorphism.

(4) Prove that for each ring $(X, +, 0, \cdot, 1)$

$$1 = 0 \quad \text{implies} \quad X = \{0\} .$$

Such a (singleton) ring is called *trivial*.

(5) Prove that each morphism in ***Fld***,

$$f\colon (X, +, 0, \cdot, 1) \to (Y, +, 0, \cdot, 1)$$

is either one-to-one or constant; the latter case occurs iff $(Y, +, 0, \cdot, 1)$ is the trivial field. Hint: If $x_1 - x_2 \neq 0$ then the (multiplicative) inverse to $x = x_1 - x_2$ exists. Then $x^{-1} \cdot (x_1 - x_2) = 1$ implies $f(x)^{-1} \cdot f(x_1 - x_2) = 1$. Thus, $f(x_1) - f(x_2) = 0$ implies $0 = f(x) \cdot (f(x_1) - f(x_2)) = 1$.

d. The constructs ***Topc*** and ***Comp***. The full subconstruct of ***Top***, the objects of which are all compact spaces (respectively all compact T_2-spaces) is denoted by ***Topc*** (respectively ***Comp***).

(1) Prove that ***Comp*** is fibre-discrete. Hint: If $\mathrm{id}_X\colon (X, \alpha) \to (X, \beta)$ is continuous, then for each $M \in \alpha$ the set $X - M$ is closed, hence, compact. Thus $X - M$ is compact, hence closed, in β (see 1Ce(3), (4)). Thus, $\alpha \subseteq \beta$, and $\beta \subseteq \alpha$ is clear.

(2) Verify that the indiscrete topology is compact. Conclude that ***Topc*** does not have discrete fibres. Is ***Topc*** fibre-complete?

1E. Isomorphic Constructs

1. A topological space can be defined by its open sets (see 1C5) or by its closure operator (see 1Cc). Thus, we could define a construct

$$\boldsymbol{Top}' ,$$

of topological closure operators and continuous maps, as follows. Objects are pairs

$$(X, {}^{-})$$

where $^{-}$ is a map from $\exp X$ into itself which is isotone, idempotent and additive. Morphisms from $(X, {}^{-})$ to $(Y, {}^{-'})$ are the maps

$$f\colon X \to Y$$

such that

$$f(\overline{M}) \subseteq \overline{f(M)}' \qquad \text{for each} \quad M \subseteq X .$$

The constructs ***Top*** and ***Top'*** are closely related — so closely, in fact, that we are tempted to consider them as identical. We shall make this precise.

For each topology α on a set X denote by

$$I_X(\alpha)$$

the corresponding closure operator. This defines a map

$$I_X\colon \boldsymbol{Top}[X] \to \boldsymbol{Top}'[X]$$

which is easily seen to be a bijection. The important property of these bijections I_X is that they "transport" morphisms in the following sense: given topological spaces (X, α) and (Y, β) and a map $f\colon X \to Y$, then

$$f\colon (X, \alpha) \to (Y, \beta) \quad \text{is a morphism of } \boldsymbol{Top}$$

iff

$$f\colon (X, I_X(\alpha)) \to (Y, I_Y(\beta)) \quad \text{is a morphism in } \boldsymbol{Top}'.$$

(See exercise 1Cc(4).)

2. Definition. Constructs $\mathscr{S}$ and $\mathscr{T}$ are said to be *concretely isomorphic* if there exist bijections

$$I_X\colon \mathscr{S}[X] \to \mathscr{T}[X] \qquad (X \text{ a set})$$

satisfying the following condition:

(R) given arbitrary objects (X, α) and (Y, β) of $\mathscr{S}$ and an arbitrary map $f\colon X \to Y$ then $f\colon (X, \alpha) \to (Y, \beta)$ is a morphism of $\mathscr{S}$ iff $f\colon (X, I_X(\alpha)) \to (Y, I_Y(\beta))$ is a morphism of $\mathscr{T}$.

Thus, ***Top*** and ***Top'*** are concretely isomorphic constructs.

3. Example: lattices as algebras. Let $(X, \leqq)$ be a lattice. Forming all joins (of pairs) we obtain an operation

$$\vee : X \times X \to X$$

which is clearly

(i) commutative: $x_1 \vee x_2 = x_2 \vee x_1 \quad (x_1, x_2 \in X)$;

(ii) associative: $x_1 \vee (x_2 \vee x_3) = (x_1 \vee x_2) \vee x_3 \quad (x_1, x_2, x_3 \in X)$;

(iii) idempotent: $x \vee x = x \quad (x \in X)$.

Also the meet is a commutative, associative and idempotent operation

$$\wedge : X \times X \to X .$$

These two operations are related by the so-called *absorption laws*:

$$\left.\begin{aligned} x &= x \vee (y \wedge x) \\ x &= x \wedge (y \vee x) \end{aligned}\right\} \quad \text{for each } x, y \in X .$$

All this is quite easy to verify.

Still easy, though more technical, is the verification of the converse: let $\curlyvee$ and $\curlywedge$ be binary operations on a set X such that

(*) $\curlyvee$ and $\curlywedge$ are commutative, associative and idempotent, and they satisfy the absorption laws.

Then there exists a unique order relation $\leqq$ on X such that $(X, \leqq)$ is a lattice with $\curlyvee$ and $\curlywedge$ as its join and meet, respectively. (This order is, necessarily, defined by

$$x_1 \leqq x_2 \quad \text{iff} \quad x_1 \curlyvee x_2 = x_2 \qquad (x_1, x_2 \in X) .$$

It must be verified that this is really an order and that

$$x_1 \curlyvee x_2 = x_1 \vee x_2 \quad \text{and} \quad x_1 \curlywedge x_2 = x_1 \wedge x_2 \qquad \text{for all} \quad x_1, x_2 \in X .)$$

We conclude that lattices can be considered as special algebras rather than special posets. Let us formalize this.

Observation. The construct ***Lat*** is concretely isomorphic to the construct ***Lat'*** of algebras $(X, \curlyvee, \curlywedge)$, where $\curlyvee$ and $\curlywedge$ are binary operations satisfying (*), and their homomorphisms.

Indeed, for each ordering $\leqq$ in ***Lat***$[X]$ denote by $I_X(\leqq)$ the corresponding pair $(\vee, \wedge)$ of the join and the meet operations. This defines a bijective map

$$I_X\colon \textbf{\textit{Lat}}[X] \to \textbf{\textit{Lat}}'[X] .$$

It is obvious that morphisms in ***Lat*** and ***Lat'*** correspond under these bijections.

4. Definition. A *realization* of a construct $\mathscr{S}$ in a construct $\mathscr{T}$ is a full subconstruct $\mathscr{S}'$ of $\mathscr{T}$ which is isomorphic to $\mathscr{S}$.

Thus, $\mathscr{S}$ has a realization in $\mathscr{T}$ iff there exist one-to-one maps

$$I_X\colon \mathscr{S}[X] \to \mathscr{T}[X], \quad X \text{ a set},$$

such that the condition (R) of Definition 2 holds. The full subconstruct $\mathscr{S}'$ of $\mathscr{T}$ then has as objects all the pairs $(X, I_X(\alpha))$, $\alpha \in \mathscr{S}[X]$; its morphisms are all $\mathscr{T}$-morphisms between these objects.

Example: the construct ***Pros*** of preordered sets (1Bb) has a realization in the construct ***Top*** of topological spaces. For each preorder $\leqq$ on a set X denote by $I_X(\leqq)$ the following topology on X: a set $M \subseteq X$ is open iff for $x_1, x_2 \in X$ with $x_1 \leqq x_2$ and $x_1 \in M$ we have $x_2 \in M$. This defines a map

$$I_X\colon \textbf{\textit{Pros}}[X] \to \textbf{\textit{Top}}[X] ,$$

which is cearly one-to-one: the relation $\leqq$ can be retracted from the topology $I_X(\leqq)$ since $x \leqq y$ is equivalent to $y \in \overline{\{x\}}$ for all $x, y \in X$. We shall prove that also the morphisms correspond.

Let $f\colon (X, \leqq) \to (Y, \leqq)$ be an order-preserving map. Then $f\colon (X, I_X(\leqq)) \to (Y, I_Y(\leqq))$ is continuous, i.e., for each open set $M \subseteq Y$ the set $f^{-1}(M)$ is open. (If $x_1 \leqq x_2$ and $x_1 \in f^{-1}(M)$, then $f(x_1) \leqq f(x_2)$ and $f(x_1) \in M$ which implies $f(x_2) \in M$, i.e., $x_2 \in f^{-1}(M)$.)

Let $f\colon (X, I_X(\leqq)) \to (Y, I_Y(\leqq))$ be a continuous map. Then $x_1 \leqq x_2$ implies $x_2 \in \overline{\{x_1\}}$, hence $f(x_2) \in f(\overline{\{x_1\}}) \subseteq \overline{\{f(x_1)\}}$ (see 1Cc); in other words, $f(x_1) \leqq f(x_2)$.

Remarks. (i) Note that, for each preorder $\leqq$, the topology $I_X(\leqq)$ has the following special property: the intersection of an arbitrary collection of open sets is open. Such topological spaces are called *quasi-discrete*. It is rather easy to prove that, conversely, every quasi-discrete topology α has the form $\alpha = I_X(\leqq)$ for the following preorder: $x_1 \leqq x_2$ iff $x_2 \in \overline{\{x_1\}}$. Thus, the realization of ***Pros*** in ***Top*** is the full subconstruct of all quasi-discrete spaces.

(ii) The construct ***Pos*** of posets has a realization in ***Top***$_0$ (1Cd(1)): given a preorder $\leqq$ then the topology $I_X(\leqq)$ is T_0 iff $\leqq$ is an order.

5. We conclude this section with an observation explaining the role of transportable (1C7) constructs among all constructs. The reader who finds these considerations too abstract can skip this part without breaking the continuity of the text.

Let us call two structures α and β on the same set X *equivalent* if $\alpha \leqq \beta$ and $\beta \leqq \alpha$; in other words, if

$$\mathrm{id}_X\colon (X, \alpha) \to (X, \beta)$$

is an isomorphism. We write

$$\alpha \equiv \beta .$$

In a transportable construct, $\alpha \equiv \beta$ implies $\alpha = \beta$. Indeed, the bijection id_X transports α to β; since the transport is unique and $\mathrm{id}_X\colon (X, \alpha) \to (X, \alpha)$ is also an isomorphism, this implies $\alpha = \beta$.

In the construct ***Mtc*** (1C8), two metrics are equivalent iff they induce the same topology.

Let us say that a construct $\mathscr{S}$ is *multi-transportable* if for each object (X, α) and each bijection $f\colon X \to Y$ there exists a structure $\beta \in \mathscr{S}[Y]$, not necessarily unique, such that

$$f\colon (X, \alpha) \to (Y, \beta)$$

is an isomorphism. Note that given another structure $\beta' \in \mathscr{S}[Y]$ with the same property, then

$$\beta \equiv \beta'.$$

For, both $f^{-1}\colon (Y, \beta) \to (X, \alpha)$ and $f\colon (X, \alpha) \to (Y, \beta')$ are isomorphisms and hence, so is

$$\mathrm{id}_Y = f \cdot f^{-1}\colon (Y, \beta) \to (Y, \beta').$$

(Conversely, for each structure β', equivalent to β, it is clear that $f\colon (X, \alpha) \to (Y, \beta')$ is an isomorphism.)

Let us call two constructs $\mathscr{S}$ and $\mathscr{T}$ *nearly isomorphic* if there exist maps

$$I_X\colon \mathscr{S}[X] \to \mathscr{T}[X], \qquad X \text{ a set},$$

which satisfy condition (R) of Definition 1E2 and are "nearly" bijective in the following sense:

(i) if $I_X(\alpha_1) = I_X(\alpha_2)$ then $\alpha_1 \equiv \alpha_2$, for each X and $\alpha_1, \alpha_2 \in \mathscr{S}[X]$;
(ii) for each X and each $\beta \in \mathscr{T}[X]$ there exists $\alpha \in \mathscr{S}[X]$ with $\beta \equiv I_X(\alpha)$.

Proposition. Each multi-transportable construct $\mathscr{S}$ has a full subconstruct $\mathscr{T}$ such that $\mathscr{S}$ and $\mathscr{T}$ are nearly isomorphic and $\mathscr{T}$ is transportable.

Proof. For each set X, the relation $\equiv$ is clearly an equivalence relation on $\mathscr{S}[X]$. By the axiom of choice (1A3) there exists a choice class $\mathscr{T}[X]$ for $\equiv$. Finding choice classes for all sets, we obtain a full subconstruct $\mathscr{T}$ of $\mathscr{S}$: its objects are the pairs (X, α) with $\alpha \in \mathscr{T}[X]$; its morphisms are, necessarily, all $\mathscr{S}$-morphisms.

For each set X let $I_X\colon \mathscr{S}[X] \to \mathscr{T}[X]$ be the canonical map, assigning to each $\alpha \in \mathscr{S}[X]$ the unique structure $\beta = I_X(\alpha)$ such that $\alpha \equiv \beta$ and $\beta \in \mathscr{T}[X]$. Then I_X is onto and "nearly" one-to-one. Let $f\colon (X, \alpha) \to (Y, \beta)$ be a morphism in $\mathscr{S}$. Then $f\colon (X, I_X(\alpha)) \to (Y, I_Y(\beta))$ is a morphism (in $\mathscr{S}$ or $\mathscr{T}$) because it is composed of the following three morphisms: $\mathrm{id}_X\colon (X, I_X(\alpha)) \to (X, \alpha)$, $f\colon (X, \alpha) \to (Y, \beta)$, and $\mathrm{id}_Y\colon (Y, \beta) \to (Y, I_Y(\beta))$. Analogously in the converse direction.

Finally, $\mathscr{T}$ is transportable. For each object (X, α) of $\mathscr{T}$ and each bijection $f\colon X \to Y$ there exists $\beta \in \mathscr{S}[X]$ such that $f\colon (X, \alpha) \to (Y, \beta)$ is an isomorphism in $\mathscr{S}$ (because $\alpha \in \mathscr{S}[X]$ and $\mathscr{S}$ is multi-transportable). Let β' be the unique structure in $\mathscr{T}[Y]$ with $\beta \equiv \beta'$. Then $f\colon (X, \alpha) \to (Y, \beta')$ is an isomorphism in $\mathscr{T}$. Since in $\mathscr{T}$ equivalent structures are equal, the uniqueness of β' follows. □

Example. The construct ***Mtc*** of metric spaces is nearly isomorphic to the construct ***Topm*** of metrizable spaces. Consider the maps

$$I_X\colon \mathbf{Mtc}[X] \to \mathbf{Topm}[X]$$

assigning to each metric α the topology $I_X(\alpha) = \bar{\alpha}$ induced by α.

6. Definition. A *skeleton* of a construct $\mathscr{S}$ is a full subconstruct $\mathscr{S}_0$ of $\mathscr{S}$ such that for each object A in $\mathscr{S}$ there exists precisely one object A_0 in $\mathscr{S}_0$ isomorphic with A (in $\mathscr{S}$). Two constructs $\mathscr{S}$ and $\mathscr{T}$ are said to be *concretely equivalent* provided that $\mathscr{S}$ has a skeleton $\mathscr{S}_0$ and $\mathscr{T}$ has a skeleton $\mathscr{T}_0$ such that $\mathscr{S}_0$ and $\mathscr{T}_0$ are concretely isomorphic.

Remark. Each construct has a skeleton. This follows from the axiom of choice: a skeleton of $\mathscr{S}$ is nothing else than a choice class of the equivalence relation $\cong$ (Remark 1C3) together with all $\mathscr{S}$-morphisms.

Examples. (i) *Cardinals* are sets which form a skeleton of the construct ***Set***. This means that for each set X there exists a unique cardinal, denoted by

$$\mathrm{card}\, X,$$

which is isomorphic to X. Thus, two sets X and Y are isomorphic iff $\mathrm{card}\, X = \mathrm{card}\, Y$. For finite sets this means that X has the same number of elements as Y. The usual choice of finite cardinals are the natural numbers,

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}; \quad \text{etc}.$$

The statement

$$\mathrm{card}\, X = 2$$

means that X is isomorphic to $2 = \{0, 1\}$, i.e., that X has precisely two elements.

All countable infinite sets have the same cardinal; the usual choice is the set of all natural numbers, denoted by "aleph zero":

$$\aleph_0 = \{0, 1, 2, \dots\}.$$

(ii) *Euclidean vector spaces*. Denote by $\mathbf{Vect}_0$ the full subconstruct of ***Vect*** formed by finite-dimensional vector spaces. All Euclidean spaces

$$(\mathbb{R}^n, +, \cdot) \qquad n = 0, 1, 2, \dots$$

form a skeleton of $\mathbf{Vect}_0$: each vector space A of dimension n is isomorphic to $(\mathbb{R}^n, +, \cdot)$, and two distinct Euclidean spaces are non-isomorphic.

Proposition. Each construct is concretely equivalent to a transportable construct.

Proof. Let $\mathscr{S}$ be an arbitrary construct. We shall exhibit a multi-transportable construct $\mathscr{T}$, concretely equivalent to $\mathscr{S}$. Then we shall use the preceding proposition: $\mathscr{T}$ is nearly isomorphic to a transportable construct $\mathscr{T}'$. It is obvious that $\mathscr{S}$ and $\mathscr{T}'$ are also concretely equivalent, and this will conclude the proof.

The definition of $\mathscr{T}$. Its objects are quadruples

$$(X, \hat{X}, \alpha, p)$$

where X is a set, $(\hat{X}, \alpha)$ is an object of $\mathscr{S}$ and $\hat{p}$: $X \to X$ is a bijection. Its morphisms from $(X, \hat{X}, \alpha, p)$ to $(Y, \hat{Y}, \beta, q)$ are maps f: $X \to Y$ such that

$$q^{-1}.f.p: (\hat{X}, \alpha) \to (\hat{Y}, \beta)$$

is a morphism of $\mathscr{S}$. Let us verify the axiom of composition. Given morphisms

$$f: (X, \hat{X}, \alpha, p) \to (Y, \hat{Y}, \beta, q) \quad \text{and} \quad g: (Y, \hat{Y}, \beta, q) \to (Z, \hat{Z}, \gamma, r)$$

then $q^{-1}.f.p$ and $r^{-1}.g.q$ are morphisms in $\mathscr{S}$, hence, also

$$(r^{-1}.g.q).(q^{-1}.f.p) = r^{-1}.(g.f).p: (\hat{X}, \alpha) \to (\hat{Z}, \gamma)$$

is a morphism in $\mathscr{S}$. We see that $\mathscr{T}$ is a well-defined construct.

$\mathscr{T}$ is multi-transportable: for each object $(X, \hat{X}, \alpha, p)$ and each bijection f: $X \to Y$ we obtain a new object $(Y, \hat{X}, \alpha, f.p)$, and clearly,

$$f: (X, \hat{X}, \alpha, p) \to (Y, \hat{X}, \alpha, f.p)$$

is an isomorphism.

$\mathscr{T}$ is concretely equivalent to $\mathscr{S}$. Let $\mathscr{S}_0$ be an arbitrary skeleton of $\mathscr{S}$. Denote by $\mathscr{T}_0$ the full subconstruct of $\mathscr{T}$, the objects of which are all the quadruples

$$(X, X, \alpha, 1_X)$$

with (X, α) in $\mathscr{S}_0$. Then $\mathscr{S}_0$ and $\mathscr{T}_0$ are obviously conretely isomorphic; hence, it suffices to show that $\mathscr{T}_0$ is a skeleton of $\mathscr{T}$. First, no two objects of $\mathscr{T}_0$ are isomorphic (since this holds for $\mathscr{S}_0$). Next, for each object

$$(X, \hat{X}, \alpha, p)$$

in $\mathscr{T}$ there exists an object (Y, β) in $\mathscr{S}_0$, isomorphic to $(\hat{X}, \alpha)$; let f: $(Y, \beta) \to (\hat{X}, \alpha)$ be an isomorphism. Then

$$p.f: (Y, Y, \beta, 1_Y) \to (X, \hat{X}, \alpha, p)$$

is an isomorphism in $\mathscr{T}$ (because both $p^{-1}.(p.f).1_Y$ and $1_Y^{-1}.(p.f)^{-1}.p$ are morphisms in $\mathscr{S}$). □

7. Example: One-object constructs. Let $\mathscr{S}$ be a construct which has precisely one object $A = (X, \alpha)$. The symbol α is somewhat superfluous: since the structure is unique, it is only important to know what the set X is and what the morphisms

are. Each morphism is a transformation of X (see 1Da). By the axioms of composition and units,

$$\hom(A, A) \subseteq X^X$$

is a transformation monoid.

Thus, we can identify transformation monoids and one-object constructs.

For each such construct $\mathscr{S}$ we obtain a transportable construct $\mathscr{T}$ concretely equivalent to $\mathscr{S}$ as follows. Its objects are (Y, α), where Y is a set and α: $Y \to X$ is a bijection. (Thus, all objects in $\mathscr{T}$ are on sets isomorphic to X.) The morphisms from (Y, α) to (Y', α') are those maps

$$f: Y \to Y'$$

for which $\alpha'.f.\alpha^{-1}$: $X \to X$ is an element of the transformation monoid.

1F. Subobjects and Generation

1. Each subset of a poset is also ordered: by the restriction of the given order. On the other hand, subsets of a vector space need not carry a structure of a subspace. We are able to formulate a general concept of subobject using morphisms.

Let Y be a subset of a set X. We define the *inclusion map*

$$v: Y \to X$$

by $v(y) = y$ for each $y \in Y$.

If X is ordered by a relation $\leqq$, denote by $\leqq'$ its restriction to Y: $y_1 \leqq' y_2$ in Y iff $y_1 \leqq y_2$ in X (for all $y_1, y_2 \in Y$). Then

(1) v: $(Y, \leqq') \to (X, \leqq)$ is order-preserving.

Note that, however, this condition alone does not determine the order $\leqq'$: if $\preceq$ denotes the discrete order then v: $(Y, \preceq) \to (X, \leqq)$ is also order-preserving. The order $\leqq'$ is determined by the following property (easy to verify):

(2) for each poset $(T, \preceq)$ and each map h: $T \to Y$ such that $v.h$: $(T, \preceq) \to (X, \leqq)$ is a morphism, h: $(T, \preceq) \to (Y, \leqq')$ is also a morphism.

Conditions (1) and (2) do determine $\leqq'$ since (1) is fulfilled by all orders finer than $\leqq'$, and (2) is fulfilled by all orders coarser than $\leqq'$.

2. Definition. Let (X, α) be an object and let Y be a subset of X. An object (Y, β) is a *subobject* of (X, α) if the inclusion map v: $Y \to X$ fulfils the following:

(1) v: $(Y, \beta) \to (X, \alpha)$ is a morphism;

(2) for each object (T, δ) and each map h: $T \to Y$ such that $v.h$: $(T, \delta) \to (X, \alpha)$ is a morphism, also h: $(T, \delta) \to (Y, \beta)$ is a morphism.

$$\begin{tikzcd} (T,\delta) \arrow[d, "h"] & \\ (Y,\beta) \arrow[r, "v"] & (X,\alpha) \end{tikzcd}$$

$r: Y \to X;\ r(y) = y\ \forall y \in Y$

$(Y, \le')$ is a subobj. of $(X, \le)$ iff

(*) $\quad x \le' y \iff x \le y \iff r(x) \le r(y)$ for all $x, y \in Y$

1° $\quad r: (Y, \le') \to (X, \le)$ is a morphism since $x \le' y \Rightarrow r(x) \le r(y)$

$* \Rightarrow (1), (2)$

2° $\quad (Z, \preceq) \xrightarrow{h} (Y, \le') \xrightarrow{r} (X, \le)$

$r.h: (Z, \preceq) \to (X, \le)$ a morph.:

$z_1 \preceq z_2 \Rightarrow r(h(z_1)) \le r(h(z_2)) \Rightarrow h(z_1) \le' h(z_2)$

Conversely: $(1),(2) \Rightarrow (*)$.

Since $r: (Y, \le') \to (X, \le)$ is a morph., we have

$x \le' y \Rightarrow r(x) \le r(y) \iff x \le y$

Let $x \le y$

$r.h$ is a morphism, so h is a morphism too.

So, $x = h(a) \le' h(b) = y$.

$\mathrm{id}_2 : 2 \to 2$

Examples. (i) ***Top***: for each topological space (X, α) and each set $Y \subseteq X$ the following is clearly a topology on Y:

$$\alpha' = \{M \cap Y;\ M \in \alpha\}.$$

Then (Y, α') is a subobject of (X, α), called the (topological) *subspace*.

Proof. (1) $v\colon (Y, \alpha') \to (X, \beta)$ is a continuous map because

$$v^{-1}(M) = M \cap Y \in \alpha' \qquad \text{for each} \quad M \in \alpha.$$

(2) Let $v.h\colon (T, \delta) \to (X, \alpha)$ be a continuous map. Then for each $M \cap Y \in \alpha'$, where $M \in \alpha$, we have

$$h^{-1}(M \cap Y) = h^{-1}(v^{-1}(M)) = (v.h)^{-1}(M) \in \delta.$$

Hence, $h\colon (T, \delta) \to (Y, \alpha')$ is continuous.

(ii) ***Comp***: for each compact T_2-space (X, α) and each closed subset $Y \subseteq X$ the topological subspace (Y, α') is also compact T_2. Then (Y, α') is a subobject of (X, α) in ***Comp***; the proof is as in (i).

Conversely, if (Y, α') is a subobject of (X, α) in ***Comp*** then $Y \subseteq X$ is a closed subset – see 1Ce(3).

For example, the interval $[0, 1]$ with the Euclidean topology ϱ is compact, T_2. The interval $(0, 1)$ does not carry a structure of a subobject of $[0, 1]$, i.e., there is no compact T_2 topology α on $(0, 1)$ such that the space $((0, 1), \alpha)$ is a subobject of $([0, 1], \varrho)$ in ***Comp***.

(iii) ***Met***: for each metric space (X, α) and each set $Y \subseteq X$, denote by α' the restriction of the metric α to Y. Then (Y, α') is a subobject of (X, α) in ***Met***. The proof is analogous to that in (i).

Remark. A construct is said to be *hereditary* if for each object (X, α) and each set $Y \subseteq X$ there exist a subobject (Y, β) of (X, α). Thus, ***Pos***, ***Top*** and ***Met*** are examples of hereditary constructs, while ***Comp*** is not hereditary.

3. Observation. Let (Y, β) be a subobject of (X, α). Then β is the coarsest of all the structures γ on Y such that $v\colon (Y, \gamma) \to (X, \alpha)$ is a morphism.

Proof. Since

$$\mathrm{id}_Y . v = v\colon (Y, \gamma) \to (X, \alpha)$$

is a morphism, the definition of a subobject implies that also

$$\mathrm{id}_Y\colon (Y, \gamma) \to (Y, \beta)$$

is a morphism. That is, β is coarser than γ. □

Corollary. In each transportable construct, the subset $Y \subseteq X$ determines the subobject (Y, β) of (X, α): if (Y, β') is also a subobject of (X, α) then $\beta = \beta'$.

Remark: It is usual to call a subset $Y \subseteq X$ a subobject of (X, α) if there exists

a structure β such that (Y, β) is a subobject of (X, α). This leads to no contradiction in transportable constructs – this is the message of the preceding corollary. For example, we can say that the subobjects in ***Comp*** are precisely the closed subsets.

The algebraic constructs have discrete fibres, as we have seen in the section 1D. Therefore, subobjects can be defined by the condition that the inclusion v be a homomorphism alone. (By the observation above, this determines the structure β.)

Examples. (i) ***Vect***: a subobject, or a (vector) subspace, of a vector space $(X, +, \cdot)$ is a subset $Y \subseteq X$ which can carry a structure of a vector space $(Y, +', \cdot')$ such that the inclusion map is linear. The latter means that $+'$ is just the restriction of $+$:

$$y_1 +' y_2 = v(y_1 +' y_2) = v(y_1) + v(y_2) = y_1 + y_2 \qquad (\text{for all } y_1, y_2 \in Y).$$

Analogously, $\cdot'$ is just the restriction of $\cdot$.

Therefore, a subspace of a vector space $(X, +, \cdot)$ is a subset $Y \subseteq X$, closed under the addition,

$$y_1 + y_2 \in Y \quad \text{whenever} \quad y_1, y_2 \in Y,$$

and the scalar multiplication,

$$r \cdot y \in Y \quad \text{whenever} \quad y \in Y \qquad (\text{for each } r \in \mathbb{R}).$$

For example, the subspaces of the two-dimensional Euclidean space $(\mathbb{R}^2, +, \cdot)$ are 1. all the lines going through the origin, 2. the trivial subspace $\{(0, 0)\}$ and 3. $\mathbb{R}^2$ itself.

(ii) A sublattice, i.e., a subobject in the construct ***Lat***, of a lattice $(X, \leq)$ is a subset $Y \subseteq X$, closed under joins and meets, in the sense that for all $y_1, y_2 \in Y$ we have

$$y_1 \vee y_2 \in Y \quad \text{and} \quad y_1 \wedge y_2 \in Y.$$

For example, in the lattice $(\mathbb{Z}^+, \preceq)$ (1B7) the set $Y \subseteq \mathbb{Z}^+$ of all even numbers is a sublattice: given two even numbers y_1 and y_2, their least common multiple $y_1 \vee y_2$ and their greatest common divisor $y_1 \wedge y_2$ are also even. In contrast, the set $P \subseteq \mathbb{Z}^+$ of all primes is not a sublattice – in fact, the order $\preceq$ on P is discrete!

(iii) The set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is clearly a submonoid (a subobject in the construct ***Mon***) of the additive monoid of real numbers $(\mathbb{R}, +, 0)$.

Note that, however, $\mathbb{N}$ fails to be a subgroup (a subobject in ***Grp***); indeed, $(\mathbb{N}, +, 0)$ is not even a group.

In general, a submonoid of a monoid $(X, \cdot, e)$ is a set $Y \subseteq X$ containing e and closed under the operation $\cdot$, i.e., $y_1 \cdot y_2 \in Y$ whenever $y_1, y_2 \in Y$. Whereas, a subgroup of a group $(X, \cdot, e)$ is a submonoid $Y \subseteq X$, which is also closed under the inverse-element operation, i.e., such that

$$y \in Y \quad \text{implies} \quad y^{-1} \in Y.$$

4. Definition. A construct is said to *have intersections* if the intersection of any collection of subobjects of a certain object is also a subobject. More formally, if

for each object (X, α) and each collection (Y_i, β_i), $i \in I$, of its subobjects the set $Y = \bigcap_{i \in I} Y_i$ also carries a structure of a subobject of (X, α).

This property is common to most of the constructs encountered in mathematics. Each hereditary construct (e.g., ***Top***, ***Met***, ***Gra***, ***Tos***) has intersections, of course. Let us mention some other examples.

Examples. (i) Algebraic constructs have intersections. For example, if Y_i, $i \in I$ are sublattices of a lattice $(X, \leqq)$ then so is $Y = \bigcap Y_i$. Given $y_1, y_2 \in Y$, then for all $i \in I$ we have $y_1, y_2 \in Y_i$, hence, $y_1 \vee y_2 \in Y_i$ and $y_1 \wedge y_2 \in Y_i$. This implies

$$y_1 \vee y_2 \in Y \quad \text{and} \quad y_1 \wedge y_2 \in Y.$$

Thus, ***Lat*** has intersections. It can be similarly shown that ***Clat***, ***Grd***, ***Mon***, etc. have intersections.

(ii) The construct ***Comp*** of compact Hausdorff spaces has intersections: the subobjects are precisely the closed subsets, and an intersection of closed sets is always closed (because a union of open sets is open by the definition of a topology).

Remark. The construct ***Topc*** of compact topological spaces fails to have intersections. Consider the following space (X, α):

$$X = \{\infty_1, \infty_2\} \cup \{0, 1, 2, \ldots\};$$

$$\alpha = \{U \subseteq X;\ \text{either} \quad X - U \quad \text{is finite or} \quad \{\infty_1, \infty_2\} \cap U = \emptyset\}.$$

Then (X, α) is easily seen to be compact: if U_i, $i \in I$, is its open cover then ∞_1 is an element of U_{i_0} for some $i_0 \in I$; then $X - U_{i_0}$ is a finite set, which can be covered by U_i, $i \in J$, for some finite set $J \subseteq I$, hence,

$$U_i, \quad i \in J \cup \{i_0\}$$

is a finite cover of X. It is evident that

$$Y_1 = X - \{\infty_1\} \quad \text{and} \quad Y_2 = X - \{\infty_2\}$$

are also compact subsets. Thus, the subspaces (Y_1, β_1) and (Y_2, β_2) of (X, α) are subobjects in ***Topc***.

The set

$$Y = Y_1 \cap Y_2 = \{0, 1, 2, \ldots\}$$

is not a subobject of (X, α) in ***Topc***. Indeed, the induced topology $\alpha' = \{M \cap Y;\ M \in \alpha\}$ is discrete, $\alpha' = \exp Y$. If γ is a topology such that (Y, γ) is a subobject of (X, α) then the fact that the incusion map

$$v\colon (Y, \gamma) \to (X, \alpha),$$

is continuous clearly implies $\alpha' \subseteq \gamma$, i.e., $\alpha' = \gamma$. But (Y, α') is not compact since the open cover $\{0\}, \{1\}, \{2\}, \ldots$ has no finite subcover. Therefore, γ does not exist.

5. Remark. Let (X, α) be an object of a construct with intersections. For each set $M \subseteq X$ there exists the least subobject Y of (X, α) with $M \subseteq Y$: indeed, Y is the intersection of all subobjects of (X, α) containing M. We say that the set M *generates* the subobject Y. If $Y = X$ we call M a *set of generators* of (X, α). Thus, M is a set of generators iff no subobject of (X, α), except all of (X, α), contains M.

An object (X, α) is said to *have n generators* if there is a set $M \subseteq X$ of generators with $n = \operatorname{card} M$.

Examples. (i) In the construct ***Mon*** of monoids, the additive monoid of integers

$$(\mathbb{Z}, +, 0)$$

has two generators: 1 and -1. If a submonoid $Y \subseteq \mathbb{Z}$ contains both 1 and -1, it contains $2 = 1 + 1$, $-2 = (-1) + (-1)$, $3 = 1 + 1 + 1$, $-3 \ldots$; thus, $Y = \mathbb{Z}$.

(ii) In the construct ***Grp*** of groups, $(\mathbb{Z}, +, 0)$ has one generator: 1. If a subgroup $Y \subseteq \mathbb{Z}$ contains 1 then it contains also the inverse element -1, hence $Y = \mathbb{Z}$.

Note that 1 generates the submonoid $\{0, 1, 2, \ldots\}$ of $(\mathbb{Z}, +, 0)$ (in the construct ***Mon***).

(iii) A vector space has n generators iff its dimension is at most n. Each set M generates the subspace of all linear combinations of elements of M (the linear span of M).

(iv) Generation is trivial in hereditary constructs. For example, each subset M of a topological space (X, α) generates the subspace (M, α').

(v) In the construct ***Comp***, the interval $[0, 1]$ (as a subspace of $(\mathbb{R}, \varrho)$) has $\aleph_0$ generators: the set M of all rational numbers in $[0, 1]$ is dense; hence, it is a set of generators.

Exercises 1F

a. The transitivity of subobjects. Let (Y, β) be a subobject of an object (X, α).

(1) Prove that each subobject of (Y, β) is also a subobject of (X, α).

(2) Conversely, prove that each subobject (Z, γ) of (X, α) such that $Z \subseteq Y$ is also a subobject of (Y, β).

(3) In ***Comp*** this means that given a closed set $Y \subseteq X$ then a set $Z \subseteq Y$ is closed in X iff Z is closed in Y. Is it true for each topological space (not necessarily compact)?

b. Generation implies intersections. Let $\mathscr{S}$ be a construct such that for each object (X, α) and each set M the least subobject of (X, α) containing M exists. Prove that then $\mathscr{S}$ has intersections; compare Remark 1F5. Hint: if Y_i are subobjects and $M = \bigcap Y_i$ generates a subobject Y, then, necessarily, $Y \subseteq Y_i$ for each i; thus, $M = Y$.

c. Unary Σ-algebras are sets X together with a collection (using Σ as an index set) of unary operations, i.e.,

$$\alpha_\sigma: X \to X \qquad \sigma \in \Sigma .$$

We usually identify unary algebras with pairs (X, α), where X is a set and

$$\alpha: X \times \Sigma \to X$$

is a map (related to the above operations by $\alpha(x, \sigma) = \alpha_\sigma(x)$ for $x \in X$ and $\sigma \in \Sigma$). This gives rise to the construct $\boldsymbol{Un}_\Sigma$ of unary Σ-algebras and homomorphisms (for each set Σ).

(1) Verify that $\boldsymbol{Un}_\Sigma$ is a fibre-discrete construct.

(2) Using the word-monoid Σ^* (see 1D4(iii)), define $\alpha^*: X \times \Sigma^* \to X$ as follows:

$$\alpha^*(x, \emptyset) = x;$$
$$\alpha^*(x, \sigma_1\sigma_2 \ldots \sigma_n) = \alpha_{\sigma_1}(\alpha_{\sigma_2}(\ldots(\alpha_{\sigma_n}(x))\ldots)$$

for each $x \in X$ and $\sigma_1\sigma_2 \ldots \sigma_n \in \Sigma^*$. Prove that in $\boldsymbol{Un}_\Sigma$ each subset $M \subseteq X$ generates the subalgebra $\alpha^*(M \times \Sigma^*) \subseteq X$.

d. The subalgebras of integers. (1) Verify that the additive group $(\mathbb{Z}, +, 0)$ has precisely the following subgroups:

$$k\mathbb{Z} = \{kz;\ z \in \mathbb{Z}\} \qquad k = 0, 1, 2, \ldots .$$

(2) Prove that the monoids $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, \cdot, 1)$ have uncountably many submonoids. Hint: use an arbitrary set of primes as a generating set.

e. A subbase of a topological space (X, α) is a colection α_0 of its open sets such that α is the coarsest topology on X for which $\alpha_0 \subseteq \alpha$.

(1) Prove that the intervals $(-\infty, a)$ and $(a, +\infty)$ for all $a \in \mathbb{R}$ form a subbase of the line. Prove that the strips $J \times \mathbb{R}$ and $\mathbb{R} \times J$ for all open intervals $J \subseteq \mathbb{R}$ form a subbase of the plane.

(2) Let (X, α) be a topological space with a subbase α_0. Let $f: T \to X$ be a map and let δ be a topology on T with $f^{-1}(U) \in \delta$ for each $U \in \alpha_0$. Prove that then $f: (T, \delta) \to (X, \alpha)$ is continuous. Hint: verify that $\{V \subseteq X;\ f^{-1}(V) \in \delta\}$ is a topology containing α_0 and hence, also α.

f. Generation in *Comp*. Prove that each subset of a compact T_2-space generates its closure in the space. Conclude that dense sets are precisely the generating sets. Does the same hold in *Top*? Hint: subobjects are precisely the closed sets in *Comp*.

1G. Quotient Objects

1. Quotient objects of an object (X, α) are induced by equivalences on the set X (in a manner similar to the subobjects induced by subsets of X).

Let $\sim$ be an equivalence relation on a set X. For each $x \in X$ we denote by $[x]$ its *equivalence class*, i.e., the subset of X containing all points equivalent to x:

$$[x] = \{t;\ t \in X \quad \text{and} \quad x \sim t\}.$$

All these equivalence classes form a new set, called the *quotient set* of X under $\sim$ and denoted by $X/\sim$:

$$X/\sim\ = \{[x];\ x \in X\}.$$

The map

$$\varphi: X \to X/\sim$$

defined by

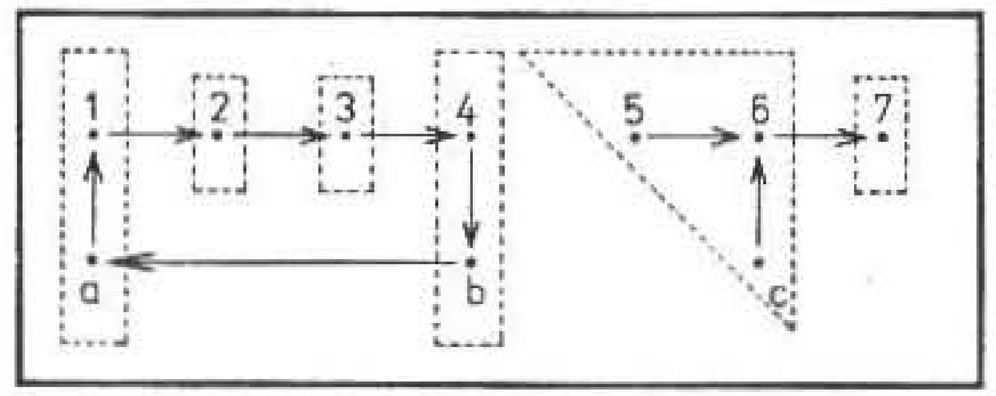
$$\varphi(x) = [x] \qquad \text{for each} \quad x \in X$$

is called the *quotient map* or the canonical map.

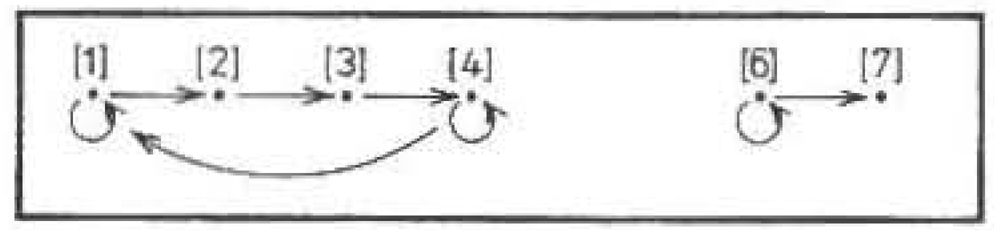
Before giving the general definition of a quotient object, let us consider the construct of graphs. For each graph (X, α) and each equivalence relation $\sim$ on X we obtain a new graph $\bar{\alpha}$ on the quotient set $X/\sim$: for arbitrary equivalence classes $t, t' \in X/\sim$

$$t\,\bar{\alpha}\,t' \quad \text{iff there exist} \quad x \in t \ \text{and}\ x' \in t' \quad \text{with} \quad x\,\alpha\,x'.$$

(Note that $x \in t$ is equivalent to $[x] = t$.) For example, consider the graph (X, α) depicted below (where an arrow from x to x' indicates that $x\,\alpha\,x'$) and the equivalence $\sim$ on X with the equivalence classes indicated by dotted lines:



(X, α)



$(X/\sim, \bar{\alpha})$

The quotient map is clearly compatible:

$$\varphi: (X, \alpha) \to (X/\sim, \bar{\alpha}).$$

Moreover, for each graph (T, δ) and each map $h: X/\sim\ \to T$ such that $h \cdot \varphi: (X, \alpha) \to (T, \delta)$ is compatible, $h: (X/\sim, \bar{\alpha}) \to (T, \delta)$ is also compatible. If $t\,\bar{\alpha}\,t'$ then there exist $x \in t$ and $x' \in t'$ with $x\,\alpha\,x'$. Then $[x] = t$, i.e., $\varphi(x) = t$; analogously, $\varphi(x') = t'$. Since $h \cdot \varphi$ is compatible, we get $(h \cdot \varphi(x))\ \delta\ (h \cdot \varphi(x'))$, i.e.,

$$h(t)\,\delta\,h(t').$$

2. Definition. Let (X, α) be an object, and let $\sim$ be an equivalence relation on the set X. An object $(X/\sim, \bar{\alpha})$ is called a *quotient object* of (X, α) under $\sim$ provided that

(1) $\varphi\colon (X, \alpha) \to (X/\sim, \bar{\alpha})$ is a morphism;

(2) for each object (T, δ) and each map $h\colon X/\sim \to T$ such that $h \cdot \varphi\colon (X, \alpha) \to (T, \delta)$ is a morphism, $h\colon (X/\sim, \bar{\alpha}) \to (T, \delta)$ is also a morphism.

$$\begin{tikzcd} (X,\alpha) \arrow[r, "\varphi"] & (X/\sim,\bar{\alpha}) \arrow[d, "h"] \\ & (T,\delta) \end{tikzcd}$$

Examples. (i) ***Gra***: the quotient graphs have been described above.

(ii) ***Top***: for each topological space (X, α) and each equivalence $\sim$ on X denote by $\bar{\alpha}$ the topology on $X/\sim$ in which a subset of $X/\sim$ is open iff the corresponding subset of X is open, i.e.,

$$\bar{\alpha} = \{U \subseteq X/\sim;\ \varphi^{-1}(U) \in \alpha\}.$$

Then $(X/\sim, \bar{\alpha})$ is a quotient space of (X, α). Clearly, φ is continuous. For each space (T, δ) and each map $h\colon X/\sim \to T$ with $h \cdot \varphi$ continuous, h is also continuous: $V \in \delta$ implies $(h \cdot \varphi)^{-1}(V) = \varphi^{-1}(h^{-1}(V)) \in \alpha$, i.e., $h^{-1}(V) \in \bar{\alpha}$.

For example, let $\sim$ be the equivalence on the real line $(\mathbb{R}, \varrho)$ with two classes: $(-\infty, 0)$ and $[0, +\infty)$. The quotient space has two points, $a_1 = [0, +\infty)$ and $a_2 = (-\infty, 0)$, and the topology on $\{a_1, a_2\}$ has three open sets: $\emptyset, \{a_2\}$ and $\{a_1, a_2\}$ – this is the Sierpinski space (1Cd(1)).

(iii) ***Sgr***: on the additive semigroup of integers $(\mathbb{Z}, +)$ define an equivalence $\sim$ as follows:

$$z_1 \sim z_2 \quad \text{iff} \quad z_1 - z_2 \text{ is even} \qquad (\text{for all } z_1, z_2 \in \mathbb{Z}).$$

Then $\mathbb{Z}/\sim$ has two classes: $[0]$ – the set of all even numbers, and $[1]$ – the set of all odd numbers. Define an operation $\oplus$ on $\mathbb{Z}/\sim$ as follows:

$$[0] \oplus [0] = [1] \oplus [1] = [0] \quad \text{and} \quad [0] \oplus [1] = [1] \oplus [0] = [1].$$

Note that for arbitrary $z_1, z_2 \in \mathbb{Z}$,

$$[z_1] \oplus [z_2] = [z_1 + z_2].$$

In other words,

$$\varphi\colon (\mathbb{Z}, +) \to (\mathbb{Z}/\sim, \oplus)$$

is a homomorphism. It is easy to see that $(\mathbb{Z}/\sim, \oplus)$ is a quotient semigroup of $(\mathbb{Z}, +)$.

Define another equivalence on $\mathbb{Z}$ as follows:

$$z_1 \sim z_2 \quad \text{iff either} \quad z_1 > 1,\ z_2 > 1 \quad \text{or} \quad z_1 \leqq 1,\ z_2 \leqq 1 \qquad (\text{for all } z_1, z_2 \in \mathbb{Z}).$$

Then $\mathbb{Z}/\sim$ also has two classes: $[1] = \{1, 0, -1, -2, \ldots\}$ and $[2] = \{2, 3, 4, \ldots\}$. But there exist no operation $\circ$ on $\mathbb{Z}/\sim$ such that $\varphi\colon (\mathbb{Z}, +) \to (\mathbb{Z}/\sim, \circ)$ is a homomorphism: $1 + 1 = 2$ forces us to define $[1] \circ [1] = [2]$, whereas $0 + 0 = 0$ forces us to define $[1] \circ [1] = [1]$.

Remark. A construct is said to be *cohereditary* if each equivalence on each object induces a quotient of this object. Thus, ***Top*** and ***Gra*** are cohereditary constructs.

For those constructs which are not cohereditary, for example ***Sgr***, it is important to know which equivalences do induce quotient objects.

3. Definition. A *congruence* on an object (X, α) is an equivalence $\sim$ on X such that there exists a quotient object of (X, α) under $\sim$.

Examples. (i) ***Grd***: a congruence on a groupoid $(X, \cdot)$ is an equivalence $\sim$ such that, for all x, x', y, y' in X,

$$(*) \qquad x \sim x' \quad \text{and} \quad y \sim y' \quad \text{imply} \quad x \cdot y \sim x' \cdot y'.$$

Proof: If $(*)$ holds, we can define an operation $\circ$ on the quotient set $X/\sim$ as follows:

$$[x] \circ [y] = [x \cdot y] \qquad \text{for all} \quad x, y \in X.$$

It is easy to verify that $(X/\sim, \circ)$ is a quotient groupoid of $(X, \cdot)$.

Conversely, if $(X/\sim, \circ)$ is a quotient groupoid then $\varphi\colon (X, \cdot) \to (X/\sim, \circ)$ is a homomorphism, thus,

$$[x] \circ [y] = \varphi(x) \circ \varphi(y) = \varphi(x \cdot y) = [x \cdot y]$$

(for all $x, y \in X$). If $x \sim x'$ and $y \sim y'$ then

$$[x] \circ [y] = [x'] \circ [y'] = [x' \cdot y'],$$

therefore, $x \cdot y \sim x' \cdot y'$.

(ii) ***Sgr***, ***Mon***, ***Grp***: the condition $(*)$ characterizes congruences.

(iii) ***Rng***: a congruence on a ring $(X, +, 0, \cdot, e)$ is an equivalence $\sim$ with the property $(*)$ both with respect to $+$ and with respect to $\cdot$. For instance, on the ring of integers $(\mathbb{Z}, +, 0, \cdot, 1)$ consider the equivalence $\equiv \pmod{k}$ of 1Db(2). This is a congruence because given $x, x', y, y' \in \mathbb{Z}$ such that $|x - x'|$ and $|y - y'|$ are divisible by k then

$$|(x + y) - (x' + y')| \quad \text{and} \quad |xy - x'y'|$$

are also divisible by k. The quotient ring has elements $[0], \ldots, [k-1]$ and its operations are induced by those in $\mathbb{Z}$:

$$[z_1] + [z_2] = [z_1 + z_2] \quad \text{and} \quad [z_1] \cdot [z_2] = [z_1 z_2]$$

(for all $z_1, z_2 \in \mathbb{Z}$).

4. Observation. Let $\sim$ be a congruence on an object (X, α). Then $(X/\sim, \bar{\alpha})$ is a quotient object of (X, α) under $\sim$ iff $\bar{\alpha}$ is the finest structure on $X/\sim$ for which the quotient map is a morphism

$$\varphi\colon (X, \alpha) \to (X/\sim, \bar{\alpha}).$$

Proof. Let $(X/\sim, \bar{\alpha})$ be a quotient object. If β is a structure such that $\varphi\colon (X, \alpha) \to (X/\sim, \beta)$ is a morphism then $\varphi = \mathrm{id}_{X/\sim} \cdot \varphi$ implies that

$$\mathrm{id}_{X/\sim}\colon (X/\sim, \bar{\alpha}) \to (X/\sim, \beta)$$

is a morphism. Thus, $\bar{\alpha}$ is finer than β.

Conversely, let $\bar{\alpha}$ be the finest structure with respect to the property above. Since $\sim$ is a congruence, there exists a quotient object $(X/\sim, \alpha_1)$; then $\varphi\colon (X, \alpha) \to (X/\sim, \alpha_1)$ is a morphism, thus, $\bar{\alpha} \leqq \alpha_1$. Since $\varphi = \mathrm{id}_{X/\sim} \cdot \varphi\colon (X, \alpha) \to (X/\sim, \bar{\alpha})$ is a morphism, we have $\alpha_1 \leqq \bar{\alpha}$. Hence, the structures $\bar{\alpha}$ and α_1 are equivalent. Consequently, $(X/\sim, \bar{\alpha})$ is a quotient object of (X, α). □

Corollary. In each transportable construct, the congruence $\sim$ determines the quotient object $(X/\sim, \bar{\alpha})$ of (X, α): if also $(X/\sim, \beta)$ is a quotient object, then $\beta = \bar{\alpha}$.

Examples. (i) ***Pos***: let $\sim$ be an equivalence on a poset $(X, \leqq)$. The finest relation on $X/\sim$ for which φ is compatible, is the following: for each $t_1, t_2 \in X/\sim$

$$t_1 \mathrel{\hat{\leqq}} t_2 \quad \text{iff there exist} \quad x_1 \in t_1,\ x_2 \in t_2 \quad \text{with} \quad x_1 \leqq x_2.$$

The relation $\hat{\leqq}$ is easily seen to be reflexive and transitive but it need not be antisymmetric.

The equivalence $\sim$ is a congruence on $(X, \leqq)$ iff the relation $\hat{\leqq}$ is antisymmetric; if so, then $(X/\sim, \hat{\leqq})$ is the quotient poset of $(X, \leqq)$.

Proof. Assuming $\hat{\leqq}$ is an ordering then $(X/\sim, \hat{\leqq})$ is a quotient poset — this follows from the preceding observation. Conversely, let $\sim$ be a congruence. Then there is an ordering $\leqq^*$ of $X/\sim$ such that $(X/\sim, \leqq^*)$ is a quotient poset. By the preceding observation, $\leqq^*$ is finer than $\hat{\leqq}$; it follows immediately that $\hat{\leqq}$ coincides with $\leqq^*$. Hence, $\hat{\leqq}$ is antisymmetric.

For example, let $(\mathbb{R}, \leqq)$ be the set of all real numbers with the usual ordering. The equivalence with the two classes $a_1 = [0, +\infty)$ and $a_2 = (-\infty, 0)$ is a congruence: the quotient poset is $(\{a_1, a_2\}, \hat{\leqq})$, where $a_2 \mathrel{\hat{\leqq}} a_1$. The equivalence

$$r_1 \sim r_2 \quad \text{iff the integer parts of } r_1 \text{ and } r_2 \text{ are equal}$$

is another congruence; the quotient poset is isomorphic to $(\mathbb{Z}, \leqq)$. But the equivalence

$$r_1 \sim r_2 \quad \text{iff} \quad r_1, r_2 \in [-1, 1] \quad \text{or} \quad r_1, r_2 \in \mathbb{R} - [-1, 1] \qquad (r_1, r_2 \in \mathbb{R})$$

is not a congruence: the above relation $\hat{\leqq}$ is not antisymmetric.

(ii) The construct ***Pros*** of preordered sets is cohereditary. For each preordered set $(X, \leqq)$ and each equivalence $\sim$, the above relation $\hat{\leqq}$ defines a quotient object $(X/\sim, \hat{\leqq})$ in ***Pros***.

5. An important example of an equivalence is the *kernel* of a map $f\colon X \to Y$; this is the equivalence $\sim$ on X defined by

$$x_1 \sim x_2 \quad \text{iff} \quad f(x_1) = f(x_2) \qquad (x_1, x_2 \in X).$$

While equivalences often do not induce quotient objects, the kernels of morphisms usually do.

Definition. A construct is said to *have kernels* if for each morphism $f\colon (X, \alpha) \to (Y, \beta)$ the kernel of f is a congruence on (X, α).

Example. (i) The construct ***Grd*** has kernels. Given a groupoid homomorphism $f\colon (X, \cdot) \to (Y, \circ)$ then the kernel equivalence $\sim$ is a congruence, since $x \sim x'$ and $y \sim y'$ imply

$$\begin{aligned} f(x \cdot y) &= f(x) \circ f(y) && [f \text{ is a homomorphism}],\\ &= f(x') \circ f(y') && [f(x) = f(x') \text{ and } f(y) = f(y')],\\ &= f(x' \cdot y') && [f \text{ is a homomorphism}], \end{aligned}$$

thus, $x \cdot y \sim x' \cdot y'$.

The same holds for ***Mon***, ***Grp*** and other algebraic constructs.

(ii) The construct ***Pos*** has kernels: given an order-preserving map $f\colon (X, \leqq) \to (Y, \sqsubseteq)$, then the relation $\hat{\leqq}$ of the preceding examples is antisymmetric. If $t_1 \mathrel{\hat{\leqq}} t_2$ then there exist $x_1 \in t_1$ and $x_2 \in t_2$ with $x_1 \leqq x_2$; if also $t_1 \mathrel{\hat{\geqq}} t_2$ then there exist $x_1' \in t_1$ and $x_2' \in t_2$ with $x_1' \geqq x_2'$. Then $x_1 \sim x_1'$, i.e., $f(x_1) = f(x_1')$, and $f(x_2) = f(x_2')$. Hence, $x_1 \leqq x_2$ implies

$$f(x_1) \sqsubseteq f(x_2)$$

and $x_2' \leqq x_1'$ implies

$$f(x_2) = f(x_2') \sqsubseteq f(x_1') = f(x_1);$$

therefore, $f(x_1) = f(x_2)$. In other words, $x_1 \sim x_2$; equivalently, $t_1 = t_2$.

(iii) ***Top***, ***Gra*** and all other cohereditary constructs have kernels, of course.

Remark. In a construct with kernels each morphism $f\colon (X, \alpha) \to (Y, \beta)$ can be factored as $f = f' \cdot \varphi$, where $\varphi\colon (X, \alpha) \to (T, \delta)$ is the quotient morphism and $f'\colon (T, \delta) \to (Y, \beta)$ is a one-to-one morphism. In fact, let (T, δ) be the quotient object of (X, α) under the kernel equivalence of f. Define $f'\colon X/\sim \to Y$ by

$$f'([x]) = f(x) \qquad \text{for each} \quad x \in X.$$

This is a map such that $f = f' \cdot \varphi$; since $f' \cdot \varphi\colon (X, \alpha) \to (Y, \beta)$ is a morphism, so is $f'\colon (T, \delta) \to (Y, \beta)$. Moreover, f' is one-to-one since $[x_1] \neq [x_2]$ is equivalent to $f(x_1) \neq f(x_2)$.

6. Another factorization of morphisms is possible in constructs such that the image of each morphism is a subobject. For each map

$$f\colon X \to Y$$

we denote by

$$\operatorname{im} f = f(X) \subseteq Y$$

its image.

Definition. A construct is said to *have images* if for each morphism $f\colon (X,\alpha) \to (Y,\beta)$ the set $\operatorname{im} f$ is a subobject of (Y,β).

Examples. (i) The construct ***Grd*** has images: given a groupoid homomorphism $f\colon (X,\cdot) \to (Y,\circ)$, the set $\operatorname{im} f$ is a subgroupoid of $(Y,\circ)$. For each $y_1, y_2 \in \operatorname{im} f$ there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$ and $f(x_2) = y_2$; then

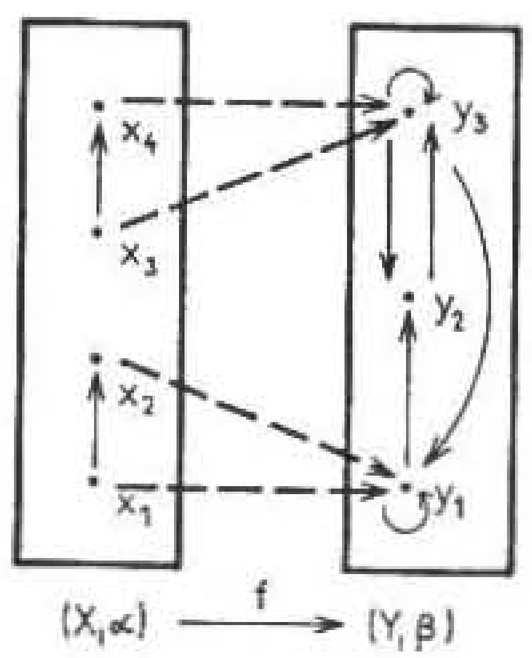
$$y_1 \circ y_2 = f(x_1) \circ f(x_2) = f(x_1 \cdot x_2) \in \operatorname{im} f.$$

Similarly with other algebraic constructs.

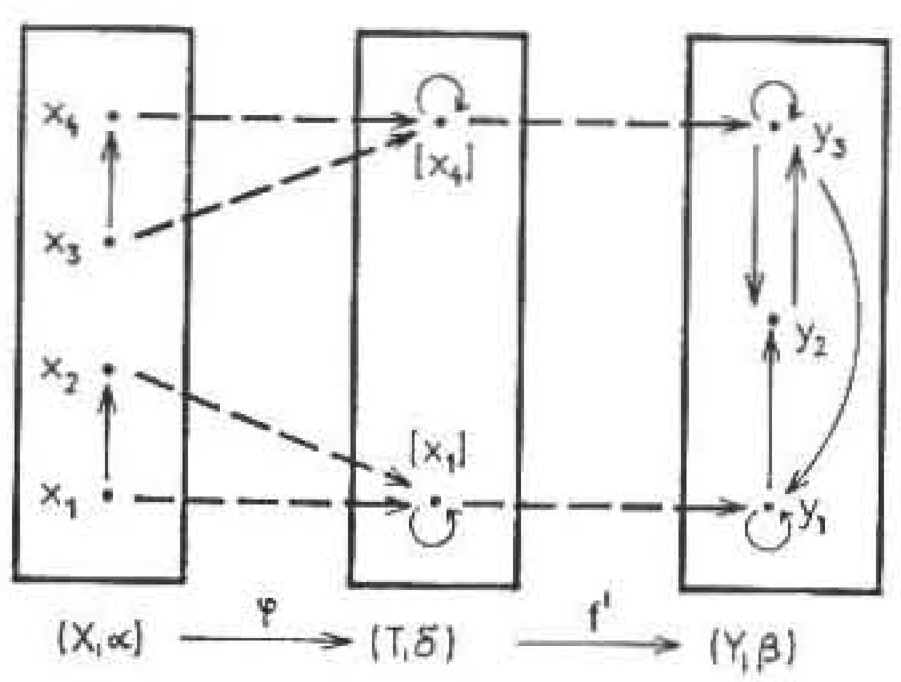
(ii) The construct ***Comp*** has images: given a continuous map $f\colon (X,\alpha) \to (Y,\beta)$ with (X,α) a compact space, then $\operatorname{im} f$ is a compact (1Ce(4)); if (Y,β) is a T_2-space then $\operatorname{im} f$ is a compact T_2-subspace.

Remark. In a construct with images, each morphism $f\colon (X,\alpha) \to (Y,\beta)$ can be factored as $f = v.\bar{f}$, where $\bar{f}\colon (X,\alpha) \to (T,\delta)$ is a morphism with $\bar{f}$ onto and $v\colon (T,\delta) \to (Y,\beta)$ is the inclusion morphism.

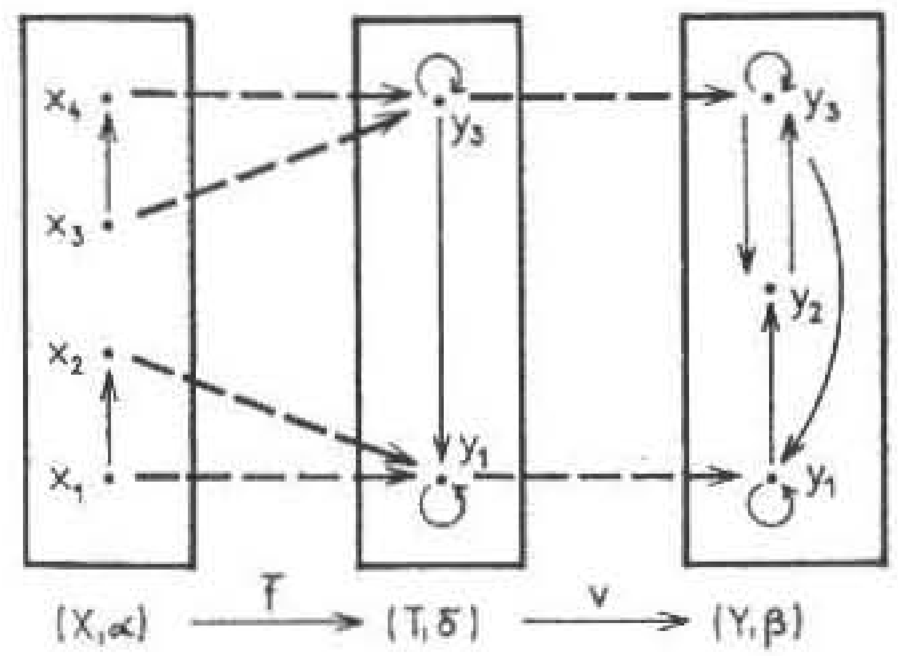
For example, the construct ***Gra*** has images and kernels. Consider the following morphism f:



The factorization $f = f'.\varphi$ is the following one:



The factorization $f = v.\bar{f}$ is the following one:



Exercises 1G

a. Congruences defined by subobjects.

(1) ***Ab***: for each subgroup Y of an Abelian group $(X, +, 0)$ define the following equivalence on X:

$$x_1 \approx_Y x_2 \quad \text{iff} \quad x_1 - x_2 \in Y \qquad (x_1, x_2 \in X).$$

Prove that $\approx_Y$ is a congruence and $Y = [0]$. Conversely, prove that for each congruence $\sim$, the class $[0] = Y$ is a subgroup such that $\approx_Y$ coincides with $\sim$.

(2) Describe all quotients of the additive group $(\mathbb{Z}, +, 0)$ of integers and prove that they are isomorphic to the groups of 1Db. Hint: see 1Fd(1).

(3) ***Vect***: prove that, analogously, the congruences on a vector space are precisely the equivalences $\approx_Y$, where Y is an (arbitrary) subspace.

(4) Describe all congruences on the two-dimensional Euclidean space $(\mathbb{R}^2, +, \cdot)$.

(5) ***Grp***: a subgroup Y of a (non-Abelian) group $(X, \cdot, e)$ is said to be *normal* if for each $y \in Y$ and $x \in X$ we have $x \cdot y \cdot x^{-1} \in Y$. Prove that the congruences on a group are precisely the equivalences $\approx_Y$, where Y is a normal subgroup.

(6) ***Rng***: a subring Y of a ring $(X, +, 0, \cdot, e)$ is called an *ideal* if for each $y \in Y$ and $x \in X$ we have $x \cdot y \in Y$. Prove that the congruences on a ring are precisely the equivalences $\approx_Y$, where Y is an ideal.

(7) Describe all congruences on the ring of integers. Hint: see Example 1G3(iii).

(8) ***Fld***: prove that no non-trivial equivalence on a field is a congruence.

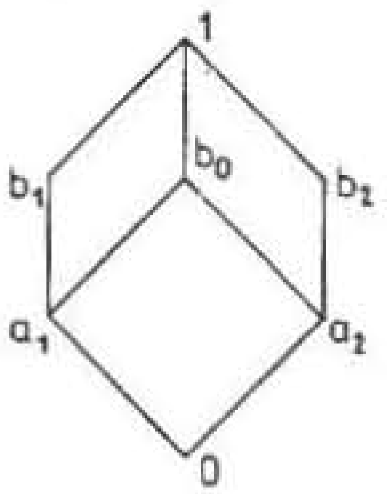
b. Congruences in $\boldsymbol{Top}_1$. Let (X,α) be a T_1-space. Prove that an equivalence $\sim$ is a congruence in $\boldsymbol{Top}_1$ iff each of its classes $[x]$ is a closed subset of X.

c. Congruences on posets and lattices.

(1) Let $(X, \leqq)$ be a lattice. Prove that an equivalence $\sim$ on X is a congruence in ***Lat*** iff for all $x, . \, , y, y' \in X$ with $x \sim x'$ and $y \sim y'$ we have

$$(x \vee y) \sim (x' \vee y') \quad \text{and} \quad (x \wedge y) \sim (x' \wedge y').$$

(2) Let $(X, \leq)$ be the poset with the following Hasse diagram



(This means that $X = \{0, a_1, a_2, b_1, b_0, b_2, 1\}$, and $\leq$ is the least ordering such that for each edge $x - y$ in this diagram with x lower than y we have $x \leq y$.)

Prove that $(X, \leq)$ is a lattice. Denote by $\sim$ the least equivalence with $b_1 \sim b_2$ (i.e., the only non-singleton class of $\sim$ is $\{b_1, b_2\}$). Prove that $\sim$ is a congruence in ***Pos*** but not a congruence in ***Lat***.

(3) Prove that in ***Clat*** a congruence on a complete lattice $(X, \leq)$ is an equivalence $\sim$ such that for each collection $(x_i, x_i') \in X \times X$, $i \in I$, of pairs of elements with $x_i \sim x_i'$ (for all $i \in I$) we have

$$\bigvee_{i \in I} x_i \sim \bigvee_{i \in I} x_i' \quad \text{and} \quad \bigwedge_{i \in I} x_i \sim \bigwedge_{i \in I} x_i' .$$

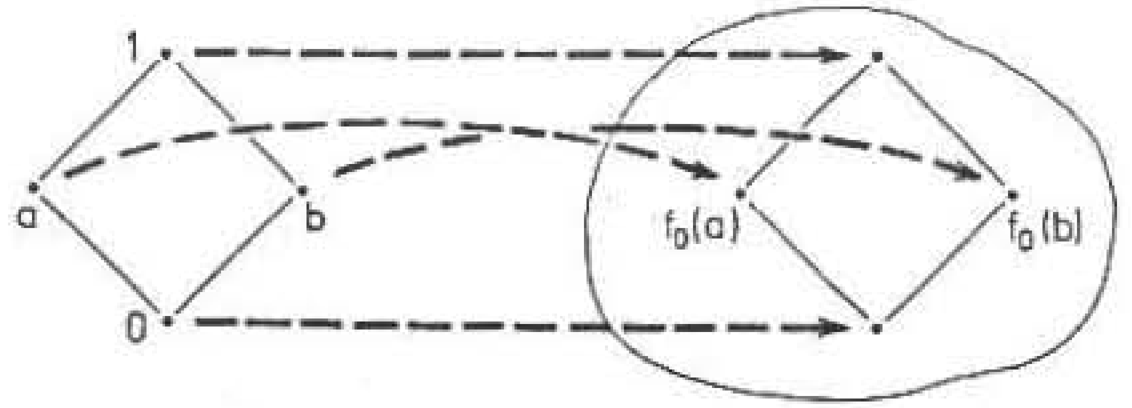
(4) Consider the complete lattice $[0, 1]$ (with the usual order). Let $\sim$ be the equivalence with the classes $\{0\}$ and $(0, 1]$. Prove that $\sim$ is not a congruence in ***Clat*** though it is a congruence in ***Lat***, where the quotient lattice is complete!

d. Factorization of morphisms. Prove that in each construct which has both images and kernels, every morphism $f: (X, \alpha) \to (Y, \beta)$ can be factored as $f = v . f^* . \varphi$ where $\varphi: (X, \alpha) \to (T, \delta)$ is the quotient morphism, $f^*: (T, \delta) \to (T', \delta')$ is a bijective morphism and $v: (T', \delta') \to (Y, \beta)$ is the inclusion morphism. Illustrate this on the example in ***Gra*** in Remark 1G6.

1H. Free Objects

1. Definition. An object (X, α) is said to be *free over a set* $M \subseteq X$ provided that for each object (Y, β) and each map $f_0: M \to Y$ there exists a unique morphism $f: (X, \alpha) \to (Y, \beta)$ extending f_0 (i.e., with $f(m) = f_0(m)$ for all $m \in M$).

Examples. (i) The four-element lattice $A = (\{0, a, b, 1\}, \leq)$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$ and a and b are incompatible, is free in ***Lat*** over $\{a, b\}$.



Let $B = (Y, \leq)$ be a lattice and let $f_0: \{a, b\} \to Y$ be a map. Since in A we have

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1,$$

the extension of f_0 to a homomorphism $f: A \to B$ must fulfil

(1) $\qquad f(0) = f(a) \wedge f(b) = f_0(a) \wedge f_0(b)$

and

(2) $\qquad f(1) = f(a) \vee f(b) = f_0(a) \vee f_0(b)$.

On the other hand, when extending f_0 to f by (1) and (2) we clearly obtain a homomorphism $f: A \to B$.

(ii) The additive monoid of natural numbers $(\mathbb{N}, +, 0)$ is the free monoid over $\{1\}$. Let $(Y, \cdot, e)$ be a monoid and let $f_0: \{1\} \to Y$ be a map: put $y_0 = f_0(1)$. Then f_0 has a unique extension to a monoid homomorphism $f: (\mathbb{N}, +, 0) \to (Y, \cdot, e)$, viz.,

$$\begin{aligned} f(0) &= e, \\ f(1) &= y_0, \\ f(2) &= f(1 + 1) = f(1) . f(1) = y_0 \cdot y_0, \\ f(3) &= f(1 + 1 + 1) = f(1) . f(1) . f(1) = y_0 \cdot y_0 \cdot y_0, \end{aligned}$$

etc..

(iii) For each set M the word-monoid $(M^*, \cdot, \emptyset)$ (see 1D4(iii)) is free over the set M, where each $m \in M$ is considered as a one-letter word. Let $(Y, \circ, e)$ be a monoid and let $f_0: M \to Y$ be a map. Then f_0 has a unique extension to a homomorphism $f: (M^*, \cdot, \emptyset) \to (Y, \circ, e)$, viz.,

$$\begin{aligned} f(\emptyset) &= e, \\ f(m_1) &= f_0(m_1) \quad \text{for each } m_1 \in M, \\ f(m_1 m_2) &= f_0(m_1) \circ f_0(m_2) \quad \text{for each } m_1, m_2 \in M, \\ f(m_1 m_2 m_3) &= f_0(m_1) \circ f_0(m_2) \circ f_0(m_3) \quad \text{for each } m_1, m_2, m_3 \in M, \end{aligned}$$

etc.

(iv) Every vector space is free in ***Vect***. In fact, if M is a basis of a vector space $(X, +, \cdot)$ then each vector $x \in X$ is a unique linear combination

$$\sum_{i=1}^{n} r_i m_i, \qquad r_i \in \mathbb{R}; \quad m_i \in M \ (i = 1, \ldots, n).$$

Let $(Y, +, \cdot)$ be another vector space, and let $f_0: M \to Y$ be a map. The unique extension to a linear map $f(X, +, \cdot) \to (Y, +, \cdot)$ is defined by

$$f(x) = \sum r_i f_0(m_i)$$

for each linear combination $x = \sum r_i m_i$ of the base vectors.

Remark. If (X, α) is a free object over $M \subseteq X$ then morphisms on (X, α) are determined by M. That is, if two morphisms

$$f, g: (X, \alpha) \to (Y, \beta)$$

fulfil

$$f(m) = g(m) \qquad \text{for all} \quad m \in M$$

then $f = g$. Denote by $f_0\colon M \to Y$ the (joint) restriction of f and g; then f and g are both extensions of f_0 to morphisms. Since the extension of f_0 is unique by definition, then $f = g$.

2. Terminology. If (X, α) is a free object over M then M is called a *set of free generators*. We also say that (X, α) is a *free object on n generators* if card $M = n$. Let us prove that this terminology is consistent with that of 1F5.

Proposition. Let (X, α) be a free object over $M \subseteq X$. Then M is a set of generators of (X, α).

Proof. We are to show that for each subobject (Y, β) of (X, α)

$$M \subseteq Y \quad \text{implies} \quad Y = X .$$

Denote the three inclusion maps as follows:

$$v\colon M \to X; \qquad v_1\colon M \to Y \quad \text{and} \quad w\colon Y \to X .$$

Thus,

$$v = w \cdot v_1 .$$

The map $v_1\colon M \to Y$ can be extended to a morphism $\bar{v}_1\colon (X, \alpha) \to (Y, \beta)$. The morphism

$$w \cdot \bar{v}_1\colon (X, \alpha) \to (X, \alpha)$$

fulfils, for each $m \in M$,

$$\begin{aligned} w \cdot \bar{v}_1(m) &= w(\bar{v}_1(m)) \\ &= w(v_1(m)) \\ &= v(m) \\ &= m . \end{aligned}$$

Thus, $w \cdot \bar{v}_1$ coincides with id_X on the set M. By the Remark above, this implies

$$w \cdot \bar{v}_1 = \mathrm{id}_X\colon (X, \alpha) \to (X, \alpha) .$$

Hence, for each $x \in X$ we have

$$x = w(\bar{v}_1(x)) \in Y, \quad (= \bar{v}_1(x))$$

which proves that $X = Y$. □

3. Remark. A free object over a set M is not uniquely determined by the set M. For example, in 1H1 we saw that $(\mathbb{N}, +, 0)$ is a free monoid over $\{1\}$, and also the word-monoid $(\{1\}^*, \cdot, \emptyset)$ is a free monoid over $\{1\}$. Note that the elements of $\{1\}^*$ are

$$\emptyset, 1, 11, 111, \ldots, 1^n = 11 \ldots 1 \ (n\text{-times}), \ldots .$$

It is clear that $(\mathbb{N}, +, 0)$ is isomorphic with $(\{1\}^*, \cdot, \emptyset)$ under the following bijection $f\colon \mathbb{N} \to \{1\}^*$:

$$f(0) = \emptyset; \; f(1) = 1; \; f(2) = 11; \; \ldots; \; f(n) = 1^n, \ldots .$$

We shall prove that this is no coincidence.

Proposition. A free object is uniquely determined up to an isomorphism by its number of free generators. That is,

(1) if A and A' are free objects on n generators then

A is isomorphic to A';

(2) if A is a free object on n generators then each object A', isomorphic to A, is also free on n generators.

Proof. (1) Let $A = (X, \alpha)$ be free over a set M and let $A' = (X', \alpha')$ be free over a set M'. If M and M' have the same cardinality, there exists a bijection $f_0\colon M \to M'$. Denote by $v\colon M \to X$ and $v'\colon M' \to X'$ the inclusion maps.

$$\begin{tikzcd} M \arrow[r, shift left, "f_0"] \arrow[d, "v"'] & M' \arrow[l, shift left, "f_0^{-1}"] \arrow[d, "v'"] \\ (X, \alpha) \arrow[r, shift left, "f"] & (X', \alpha') \arrow[l, shift left, "\bar{f}"] \end{tikzcd}$$

The map $v' \cdot f_0\colon M \to X'$ has an extension to a morphism

$$f\colon (X, \alpha) \to (X', \alpha') .$$

And the map $v \cdot f_0^{-1}\colon M' \to X$ has an extension to a morphism

$$\bar{f}\colon (X', \alpha') \to (X, \alpha) .$$

To prove that f is an isomorphism, it suffices to show that f and $\bar{f}$ are inverse to each other.

The morphism

$$\bar{f} \cdot f\colon (X, \alpha) \to (X, \alpha)$$

fulfils, for each $m \in M$,

$$\begin{aligned} \bar{f} \cdot f(m) &= \bar{f} \cdot v' \cdot f_0(m) = \\ &= v \cdot f_0^{-1} \cdot f_0(m) \\ &= v(m) \\ &= m . \end{aligned}$$

By Remark 1H1, this implies

$$\bar{f} . f = \mathrm{id}_X .$$

Analogously,

$$f . \bar{f} = \mathrm{id}_{X'} .$$

Hence, $\bar{f} = f^{-1}$ and thus, $f: A \to A'$ is an isomorphism.

(2) Let $g: (X, \alpha) \to (X', \alpha')$ be an isomorphism, and assume that (X, α) is free over $M \subseteq X$. It suffices to prove that (X', α') is free over the set

$$M' = g(M) .$$

$$\begin{tikzcd}
M \arrow[rr, shift left, "g_0"] \arrow[dd, "v"'] & & M' \arrow[ll, shift left, "g_0^{-1}"] \arrow[dd, "v'"] \arrow[dl, "f_0"] \\
& (Y, \beta) & \\
(X, \alpha) \arrow[ur, "\bar{f}"] \arrow[rr, shift left, "g"] & & (X', \alpha') \arrow[ll, shift left, "g^{-1}"]
\end{tikzcd}$$

Let (Y, β) be an object and let $f_0: M' \to Y$ be a map. We extend f_0 to a morphism. To this end, denote by

$$g_0: M \to M'$$

the domain-range restriction of the bijection g. The map $f_0 . g_0: M \to Y$ has an extension to a morphism

$$\bar{f}: (X, \alpha) \to (Y, \beta) .$$

Then the morphism

$$f = \bar{f} . g^{-1}: (X', \alpha') \to (Y, \beta)$$

is an extension of f_0: for each $m \in M'$ we have $g^{-1}(m) \in M$; thus,

$$\begin{aligned} f(m) &= \bar{f}(g^{-1}(m)) \\ &= f_0 . g_0(g^{-1}(m)) \\ &= f_0(m) . \end{aligned}$$

It remains to prove that f is unique. If $f_1: (X', \alpha') \to (Y, \beta)$ is also an extension of f_0, then the morphisms

$$f . g, f_1 . g: (X, \alpha) \to (Y, \beta)$$

clearly coincide on M and, hence, by Remark 1H1,

$$f . g = f_1 . g .$$

This implies

$$f = f . g . g^{-1} = f_1 . g . g^{-1} = f_1 . \qquad \square$$

4. Definition. A construct is said to *have free objects* if for each cardinal number n there exists a free object on n generators.

Examples. (i) The construct ***Mon*** of monoids has free objects, as we have seen in 1H1.

We shall later prove that other algebraic constructs, ***Grd***, ***Grp***, ***Lat***, etc., have free objects. Free objects in algebraic constructs are usually interesting algebras, and the investigation of their properties is an important part of modern algebra.

(ii) The construct ***Top*** has free objects: for each set X the discrete space $(X, \exp X)$ is free over X. Given a topological space (Y, β), then each map $f_0: X \to Y$ is continuous, i.e., $f_0: (X, \exp X) \to (Y, \beta)$ is a morphism.

(iii) The construct ***Pos*** has free objects: for each set X the discrete poset $(X, \leq)$ (where $x_1 \leq x_2$ is equivalent to $x_1 = x_2$) is free over X. Given a poset $(Y, \leq)$, then each map $f_0: X \to Y$ is order-preserving, $f_0: (X, \leq) \to (Y, \leq)$.

Generalizing the situation in ***Top*** and ***Pos***, we call an object (X, α) *discrete* if for each object (Y, β), all maps $f: X \to Y$ are morphisms $f: (X, \alpha) \to (Y, \beta)$. Equivalently, an object (X, α) is discrete iff it is free over all of X.

Further examples. (iv) The construct ***Met*** does not have free objects on two or more generators. If $\operatorname{card} M > 1$ and if (X, α) is a free metric space over $M \subseteq X$, consider the space $(X, 2\alpha)$: the inclusion $f_0 = v: M \to X$ has, of course, no extension to a contraction $f: (X, \alpha) \to (X, 2\alpha)$.

For each real number $k > 0$ denote by

$$\boldsymbol{Met}_k$$

the full subconstruct of ***Met***, the objects of which are the metric spaces (X, α) with diameter at most k, i.e., such that

$$\alpha(x_1, x_2) \leq k \qquad \text{for all} \quad x_1, x_2 \in X .$$

Then $\boldsymbol{Met}_k$ has free objects, in fact, discrete objects: for each set X define a metric α by

$$\alpha(x_1, x_2) = \begin{cases} k & \text{if} \quad x_1 \neq x_2 \quad (x_1, x_2 \in X) \\ 0 & \text{if} \quad x_1 = x_2 . \end{cases}$$

Then (X, α) is discrete in $\boldsymbol{Met}_k$.

(v) The construct ***Clat*** of complete lattices does not have free objects on three or more generators. The proof is beyond the scope of this book (see A. W. Hales, *Fundamenta Mathematicae* 54 (1964), 45–66).

In contrast, the construct ***Csl*** of complete semilattices has free objects. For each set M consider the poset $(\exp M, \leq)$, where $M_1 \leq M_2$ iff $M_2 \subseteq M_1$ $(M_1, M_2 \in \exp M)$. This is the free complete semilattice over

$$M' = \{\{m\};\ m \in M\} \subseteq \exp M .$$

Let $(Y, \leq)$ be a complete semilattice and let

$$f_0 \colon M' \to Y$$

be a map. For each set $M_1 \subseteq M$ we have, of course,

$$M_1 = \bigcup_{m \in M_1} \{m\};$$

thus, in the poset $(\exp M, \leq)$

$$M_1 = \bigwedge_{m \in M_1} \{m\}.$$

To extend f_0 to a homomorphism of complete semilattices, we must define

$$f(M_1) = \bigwedge_{m \in M_1} f_0(\{m\}) \qquad \text{for each} \quad M_1 \subseteq M .$$

Conversely, it is easy to check that the map $f\colon \exp M \to Y$ defined by the rule above is indeed a complete semilattice homomorphism extending f_0.

5. A special case of the free object is the *initial object*, which is an object A_0 such that for each object B there exists precisely one morphism from A_0 to B. The initial object is the free object over the void set (i.e., the free object on 0 generators): for each object $B = (Y, \beta)$ there exists precisely one map from $\emptyset$ to Y, the void map. This map can be uniquely extended to a morphism $f\colon A_0 \to B$; thus, $\hom(A_0, B)$ is a singleton set.

Examples. (i) $\emptyset$ is the initial object in ***Set***.

(ii) In constructs which have a structure α on $\emptyset$, the object $(\emptyset, \alpha)$ is usually initial. This is so in ***Gra***: the void subset of $\emptyset \times \emptyset = \emptyset$ is the unique relation on $\emptyset$. Similarly in subconstruct of ***Gra***: ***Pos*** and ***Pros***. Also in ***Top*** we have just one topology on $\emptyset$: $\alpha = \{\emptyset\}$. And the void groupoid is the initial object in ***Grd*** and ***Sgr***.

(iii) The initial vector space is the trivial space $(\{0\}, +, \cdot)$: for each vector space $(X, +, \cdot)$ the unique linear map $f\colon (\{0\}, +, \cdot) \to (X, +, \cdot)$ is defined by $f(0) = 0$.

Analogously, the singleton monoid (respectively group), is the initial object of ***Mon*** (***Grp***).

(iv) The ring of integers $(\mathbb{Z}, +, 0, \cdot, 1)$ is the initial object of ***Rng***: for each ring $(X, +, 0, \cdot, e)$ the unique ring homomorphism $f\colon (\mathbb{Z}, +, 0, \cdot, 1) \to (X, +, 0, \cdot, e)$ is defined as follows:

$$f(1) = e \quad \text{and} \quad f(0) = 0$$

(because f preserves the two nullary operations); hence

$$f(2) = f(1 + 1) = e + e, \quad f(3) = f(1 + 1 + 1) = e + e + e, \ldots$$

(because f preserves $+$) and

$$f(-1) = -e, \quad f(-2) = -(e + e), \quad f(-3) = -(e + e + e), \ldots$$

(because f preserves the inverses).

Concluding remark. Since free objects are determined only up to an isomorphism, it is often not so important to know their precise inner structure: their "universal" property is all that matters. Thus, in some situations we are mainly interested in the existence of free objects. In all the constructs we considered above, a free object on one generator exists (see Exercise a. below) and also an initial object exists. But with more generators the problem is not so easy.

One of the major achievements of the theory of structures is that a powerful criterion for the existence of free objects has been obtained. We introduce it in the next chapter.

Exercises 1H

a. The free object on one generator. Verify that in each of the constructs below the described object is free over the singleton set $\{x\}$.

(1) ***Top***, ***Met***, ***Gra***: the singleton object, i.e., the underlying set is $\{x\}$. The same holds in all full subconstructs containing this object, e.g., ***Comp*** and ***Pos***.

(2) ***Rng***: the ring of all polynomials with integer coefficients and with the indeterminate x. The operations $+$ and $\cdot$ are the usual addition and multiplication of polynomials; the nullary operations 0 and 1 are the constant polynomials. Hint: for each ring $(X, +, 0, \cdot, e)$ and each $f_0\colon \{x\} \to X$, $f_0(x) = t$, we can "evaluate" all polynomials, i.e., we can extend f_0 as follows:

$$\begin{aligned} f_0(a) &= e + e + \ldots + e \quad (a\text{-times, where } a \in \mathbb{Z}),\\ f_0(ax) &= (e + e + \ldots + e) \cdot t,\\ f_0(ax^2) &= (e + e + \ldots + e) \cdot t \cdot t, \end{aligned}$$

etc.

(3) ***Grp*** and ***Ab***: $(\mathbb{Z}, +, 0)$ is free over $x = 1$.

(4) ***Lat***: the singleton lattice $\{x\}$; ***Clat***: the three-element chain $(\{0, x, 1\}, \leq)$, where $0 \leq x \leq 1$. What about ***Csl***?

(5) ***Grd***: the groupoid $(T, \circ)$ of all formal expressions x, $x \circ x$, $x \circ (x \circ x)$, $(x \circ x) \circ x$, $(x \circ x) \circ (x \circ x)$, etc. Thus, T is the least set, containing x and such that $t_1, t_2 \in T$ implies $t_1 \circ t_2 \in T - \{x\}$, while $t_1 \circ t_2 = t_1' \circ t_2'$ iff $t_1 = t_1'$ and $t_2 = t_2'$ (for all $t_1, t_1', t_2, t_2' \in T$).

b. Free Abelian groups. For each set M denote by $(\hat{M}, +, \hat{p}^0)$ the group of all integer functions $p\colon M \to \mathbb{Z}$ of *finite support* (i.e., such that the set of all $m \in M$ with $p(m) \neq 0$ is finite) with the usual addition of functions:

$$(p + p')(m) = p(m) + p'(m) \qquad \text{for all} \quad p, p' \in \hat{M}, \quad m \in M ,$$

and with p^0 the constant function with value 0. Prove that this is the free Abelian group over $M' = \{p_m\}_{m \in M}$, where p_m is the function assigning 1 to m and 0 to all other elements of M.

c. Free semigroups are the semigroups of non-void words (see 1H1(iii)):

$$(M^* - \{\emptyset\}, \cdot).$$

Prove it.

d. Free unary algebras. For arbitrary sets Σ and M a free unary Σ-algebra over M is the algebra

$$(M \times \Sigma^*, \alpha)$$

where α is defined by $\alpha(m, \sigma_1 \dots \sigma_n; \sigma) = (m, \sigma\sigma_1 \dots \sigma_n)$ and where $m \in M$ is identified with $(m, \emptyset)$.

e. Free partial groupoids are the discrete objects of $\boldsymbol{Grd}_p$: these are the pairs $(X, \cdot)$ where $\cdot$ is nowhere defined.

f. Factors of an object A are the objects which are isomorphic to quotient objects of A. In algebraic constructs ($\boldsymbol{Mon}$, $\boldsymbol{Ab}$, $\boldsymbol{Un}_\Sigma$, etc.) prove that B is a factor of A iff there exists a surjective morphism $f\colon A \to B$. Conclude that each object is a factor of a free object.

g. Embedding. An object B *can be embedded* into an object A iff B is isomorphic to a subobject of A. Prove that in algebraic constructs this is the case iff there exists a one-to-one morphism $f\colon B \to A$.

h. The poset of all equivalences. For each set X, denote by $Eq(X)$ the set of all equivalences on X and define an ordering on $Eq(X)$ as in the fibre $\boldsymbol{Gra}[X]$, i.e., an equivalence α is smaller or equal to β if $\alpha \subseteq \beta$.

(1) Prove that the poset $(Eq(X), \leq)$ is a complete lattice; describe the least and the largest element. Hint: a set-theoretical intersection of equivalences is an equivalence.

(2) Prove that each poset $(X, \lesssim)$ can be embedded into $(Eq(X), \leq)$. Hint: for each $x \in X$ consider the equivalence with only one non-trivial class: $\{y \in X;\ y \lesssim x\}$.

(3) Prove that in $\boldsymbol{Csl}$ each complete semilattice can be embedded into the lattice of all equivalences. Hint: the embedding in the previous hint preserves meets.

Chapter 2: Initial and Final Structures

Many constructions in mathematics are of the following type: we are given objects A_i $(i \in I)$ and a set X and we create a new object on X using maps from X into the underlying sets of A_i or, conversely, maps from the underlying sets into X. For example, the Cartesian product of two objects $A_1 = (X_1, \alpha_1)$ and $A_2 = (X_2, \alpha_2)$ is created on the set $X = X_1 \times X_2$ by the projections. A subobject of an object $A_0 = (X_0, \alpha_0)$ is created by the inclusion map $v\colon X \to X_0$ (if $X \subseteq X_0$), and a quotient of A_0 is created by the quotient map $\varphi\colon X_0 \to X$ (if $X = X_0/\sim$).

The present chapter is devoted to a general investigation of such constructions. We first study initial structures, i.e., the case of maps leading from X, and particularly the Cartesian products. Then we turn to final structures, i.e., to maps leading into X. An important generalization of the concept of final object is the "semifinal object"; while initial and final objects often fail to exist, it turns out that semifinal objects exist in most of the constructs used in mathematics.

2A. Initial Structures

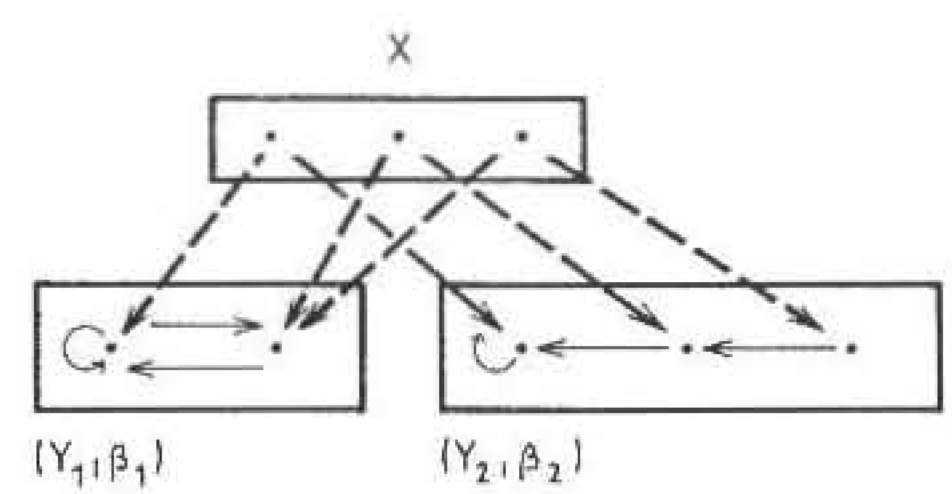
1. Let

$$f_i\colon X \to Y_i, \qquad i \in I,$$

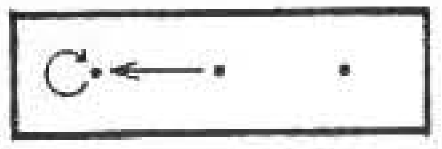
be a collection of maps with the common domain X. If on each of the sets Y_i a graph $\beta_i \subseteq Y_i \times Y_i$ is defined, then a natural graph α can be defined on X:

$$x_1 \alpha x_2 \quad \text{iff} \quad f_i(x_1)\, \beta_i\, f_i(x_2) \qquad \text{for each} \quad i \in I.$$

For example, consider the maps and graphs depicted below (where an arrow from y to y' indicates $y \beta_i y'$ in Y_i for $i = 1, 2$):



•The resulting graph on X is



Note that

(1) $f_i\colon (X, \alpha) \to (Y_i, \beta_i)$ are compatible maps for all $i \in I$. This follows immediately from the definition of α. Moreover,

(2) for each graph (T, δ) and each map $h\colon T \to X$ such that $f_i . h\colon (T, \delta) \to (Y_i, \beta_i)$ are compatible for all $i \in I$, also $h\colon (T, \delta) \to (X, \alpha)$ is compatible.

Proof: let $t_1, t_2 \in T$ be elements with $t_1 \ \delta \ t_2$; for each $i \in I$ we have $f_i(h(t_1)) \ \beta_i \ f_i(h(t_2))$. Therefore,

$$h(t_1) \ \alpha \ h(t_2);$$

hence, h is compatible.

Properties (1) and (2) determine the relation α: (1) is fulfilled by α and all the finer structures; (2) is fulfilled by α and all the coarser ones. We call α the initial structure of the given collection of maps and graphs.

2. More generally, we can introduce initial structures in an arbitrary construct $\mathscr{S}$. A *source* (in $\mathscr{S}$) on a set X is a collection (Y_i, β_i, f_i), $i \in I$, where (Y_i, β_i) are objects of $\mathscr{S}$ and $f_i\colon X \to Y_i$ are maps (for all $i \in I$). The collection is allowed to be large, i.e., I can also be a (large) class (see 1A). We usually denote sources as follows:

$$\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}.$$

Definition. An *initial structure* of a source $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ is a structure α on X such that

(1) $f_i\colon (X, \alpha) \to (Y_i, \beta_i)$ are morphisms for all $i \in I$;

(2) for each object (T, δ) and each map $h\colon T \to X$ such that all $f_i . h\colon (T, \delta) \to (Y_i, \beta_i)$ are morphisms $(i \in I)$, $h\colon (T, \delta) \to (X, \alpha)$ is also a morphism.

$$\begin{tikzcd}
 & (T, \delta) \arrow[d, "h"] & \\
 & (X, \alpha) \arrow[dl, "f_i"'] \arrow[d] \arrow[dr] & \\
(Y_i, \beta_i) & & \dots
\end{tikzcd}$$

A construct is said to be *initially complete* if each source has a unique initial structure.

Remark. Conditions (1) and (2) can be restated more compactly as follows: for each object (T, δ) and each map $h\colon T \to X$,

$$h\colon (T, \delta) \to (X, \alpha) \text{ is a morphism}$$

iff

$$f_i . h\colon (T, \delta) \to (Y_i, \beta_i) \text{ are morphisms for all } i \in I.$$

(If this holds, then put $h = \mathrm{id}_X\colon (X, \alpha) \to (X, \alpha)$ to conclude that f_i are morphisms.)

We also call (X, α) the *initial object* of the source.

Examples. (i) ***Gra*** is an initially complete construct.

(ii) ***Top*** is an initially complete construct: for each source $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ the initial topology has the following subbase (1Fe)

$$\alpha_0 = \{f_i^{-1}(M);\ i \in I \text{ and } M \in \beta_i\}.$$

In fact, if α denotes the topology with the subbase α_0 then

(1) $f_i\colon (X, \alpha) \to (Y_i, \beta_i)$ are continuous for all $i \in I$ since for each $i \in I$ and each $M \in \beta_i$ we have $f_i^{-1}(M) \in \alpha_0 \subseteq \alpha$;

(2) assuming that $f_i . h\colon (T, \delta) \to (Y_i, \beta_i)$ are continuous maps for all $i \in I$, then $h\colon (T, \delta) \to (X, \alpha)$ is also continuous. This follows easily from 1Fe: for each $f_i^{-1}(M) \in \alpha_0$ we have $h^{-1}(f_i^{-1}(M)) = (f_i . h)^{-1}(M) \in \delta$ because $f_i . h$ is continuous.

As a concrete example, consider the two projections on the plane,

$$\pi_1, \pi_2\colon \mathbb{R}^2 \to \mathbb{R},$$

and the Euclidean topology ϱ on $\mathbb{R}$ (with respect to both π_1 and π_2). The subbase α_0 consists of all the sets

$$M \times \mathbb{R} = \pi_1^{-1}(M) \quad \text{and} \quad \mathbb{R} \times M = \pi_2^{-1}(M) \qquad (M \in \varrho);$$

hence, the initial topology α is the Euclidean topology of the plane (see 1Fe).

3. Proposition. Let

$$\{X \xrightarrow{f} (Y, \beta)\}$$

be a singleton source such that f is a bijection. A structure α is initial iff

$$f\colon (X, \alpha) \to (Y, \beta)$$

is an isomorphism.

Hence, each initially complete construct is transportable.

Proof: (1) If α is initial, then f is a morphism, and since

$$\mathrm{id}_Y = f . f^{-1}\colon (Y, \beta) \to (Y, \beta)$$

is a morphism, so is $f^{-1}\colon (Y, \beta) \to (X, \alpha)$. Thus, f is an isomorphism.

(2) If f is an isomorphism and if $f . h\colon (T, \delta) \to (Y, \beta)$ is a morphism, then

$$h = f^{-1} . (f . h)\colon (T, \delta) \to (X, \alpha)$$

is also a morphism.

Observation. If α is an initial structure of a source $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ then α is the coarsest structure on X for which all f_i, $i \in I$, are morphisms.

Proof. Let $\bar{\alpha}$ be a structure such that $f_i: (X, \bar{\alpha}) \to (Y_i, \beta_i)$ are morphisms $(i \in I)$. Since all $\mathrm{id}_X . f_i: (X, \bar{\alpha}) \to (Y_i, \beta_i)$ are morphisms, $i \in I$, also

$$\mathrm{id}_X: (X, \bar{\alpha}) \to (X, \alpha)$$

is a morphism. In other words, α is coarser than $\bar{\alpha}$.

Remark. The observation above simplifies the task of determining whether an initial structure exists, and of finding it: it suffices to inspect all structures on X for which each f_i is a morphism.

Example: initial ordering. Let

$$\{X \xrightarrow{f_i} (Y_i, \leq_i)\}_{i \in I}$$

be a source in ***Pos***; does it have an initial ordering?

If $\precsim$ happens to be the initial ordering then it must be coarser than each ordering $\precsim^*$ on X for which all f_i are order-preserving, more precisely, for which

$$x \precsim^* x' \quad \text{implies} \quad f_i(x) \leq_i f_i(x') \qquad (i \in I) \qquad \text{for all} \quad x, x' \in X.$$

This condition suggests the following definition of $\precsim$:

$$x \precsim x' \quad \text{iff} \quad f_i(x) \leq_i f_i(x') \qquad (i \in I) \qquad \text{for all} \quad x, x' \in X.$$

This relation $\precsim$ is, obviously, reflexive and transitive; it need not be antisymmetric, however.

A. If $\precsim$ is antisymmetric, then it is the initial ordering.

Proof: let $(T, \leq)$ be a poset and let $h: T \to X$ be a map such that all $f_i . h: (T, \leq) \to (Y_i, \leq_i)$ are order-preserving. Then $t \leq t'$ implies $f_i(h(t)) \leq_i f_i(h(t'))$ $(i \in I)$, hence, $h(t) \precsim h(t')$. Therefore, $h: (T, \leq) \to (X, \precsim)$ is order-preserving.

B. If $\precsim$ is not antisymmetric, then the initial ordering does not exist. Proof: it suffices to show that if $\leq$ is the initial ordering then $\leq$ is a coarser relation than $\precsim$ (thus, if $\precsim$ is not antisymmetric, then $\leq$ is also not antisymmetric, which is a contradiction). Given $x_0, x_0' \in X$ with $x_0 \precsim x_0'$ we want to show that $x_0 \leq x_0'$. Define an ordering $\precsim^*$ on X as follows:

$$x \precsim^* x' \quad \text{if either} \quad x = x' \quad \text{or if} \quad x = x_0 \quad \text{and} \quad x' = x_0'$$

(for all $x, x' \in X$). Since each

$$f_i = \mathrm{id}_X . f_i: (X, \precsim^*) \to (Y_i, \leq_i), \qquad i \in I,$$

is clearly order-preserving, also $\mathrm{id}_X: (X, \precsim^*) \to (X, \leq)$ is order-preserving, i.e., $x_0 \leq x_0'$.

Observation. The above relation $\precsim$ is antisymmetric iff the source *separates points* in the sense that given $x, x' \in X$ then

$$(*) \qquad x \neq x' \quad \text{implies} \quad f_i(x) \neq f_i(x') \qquad \text{for some } i \in I.$$

If the source separates points and if $x, x' \in X$ are points such that $x \precsim x'$ as well as $x' \precsim x$, then for each $i \in I$ we have $f_i(x) \leq_i f_i(x')$ as well as $f_i(x') \leq_i f_i(x)$; hence, $f_i(x) = f_i(x')$. This implies $x = x'$.

Conversely, if the source does not separate points, then there exist distinct points $x, x' \in X$ such that $f_i(x) = f_i(x')$ for all $i \in I$. Then $x \precsim x'$ and $x' \precsim x$ and therefore, the relation $\precsim$ is not antisymmetric.

4. Definition. A construct is said to be *initially mono-complete* if each source $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ separating points (i.e., fulfilling $(*)$ above) has an initial structure.

Examples. (i) ***Pos*** is initially mono-complete.

(ii) ***Top***$_2$ (the construct of Hausdorff spaces) is initially mono-complete. Let $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ be a source of topological T_2-spaces which separates points. Then the initial topology α is also T_2: given distinct points $x, x' \in X$ there exists $i \in I$ with $f_i(x) \neq f_i(x')$; let U, V be disjoint open sets (in β_i) with $f_i(x) \in U$ and $f_i(x') \in V$. Then $f_i^{-1}(U)$ and $f_i^{-1}(V)$ are disjoint open sets (in α), and $x \in f_i^{-1}(U)$ and $y \in f_i^{-1}(V)$.

(iii) ***Met*** is not initially mono-complete.

For example, define metrics α_n on $X = \{0, 1\}$ by

$$\alpha_n(0, 1) = n, \qquad n = 1, 2, 3, \ldots.$$

Then the source

$$\{X \xrightarrow{\mathrm{id}_X} (X, \alpha_n)\}_{n=1}^{\infty}$$

does not have an initial metric. In fact, there is no metric α on X such that $\mathrm{id}_X: (X, \alpha) \to (X, \alpha_n)$ is a contraction for all n. (We can choose $n > \alpha(0, 1)$.)

(iv) For each number $k > 0$ the construct ***Met***$_k$ (of metric spaces of diameter $\leq k$) is initially mono-complete. Let $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$ be a source separating points. Define a metric on X as follows:

$$\alpha(x, x') = \bigvee_{i \in I} \beta_i(f_i(x), f_i(x')) \qquad \text{for all} \quad x, x' \in X.$$

Since $\beta_i(f_i(x), f_i(x')) \leq k$ for all i, the supremum exists, and $\alpha(x, x') \leq k$; it is also easy to see that α is indeed a metric. Each f_i is obviously a contraction. Let (T, δ) be a metric space (of diameter $\leq k$), and let $h: T \to X$ be a map such that each $f_i . h$, $i \in I$, is a contraction. Thus, for all $t, t' \in T$ we have

$$\beta_i(f_i(h(t)), f_i(h(t'))) \leq \delta(t, t') \qquad \text{for each} \quad i \in I,$$

which implies

$$\alpha(h(t), h(t')) = \bigvee_{i \in I} \beta_i(f_i(h(t)), f_i(h(t'))) \leq \delta(t, t').$$

Hence, h is a contraction.

5. Observation. Let (X,α) be an object. A subset $Y \subseteq X$ is a subobject of (X,α) iff the singleton source of the inclusion map (1F1)

$$\{Y \xrightarrow{v} (X,\alpha)\}$$

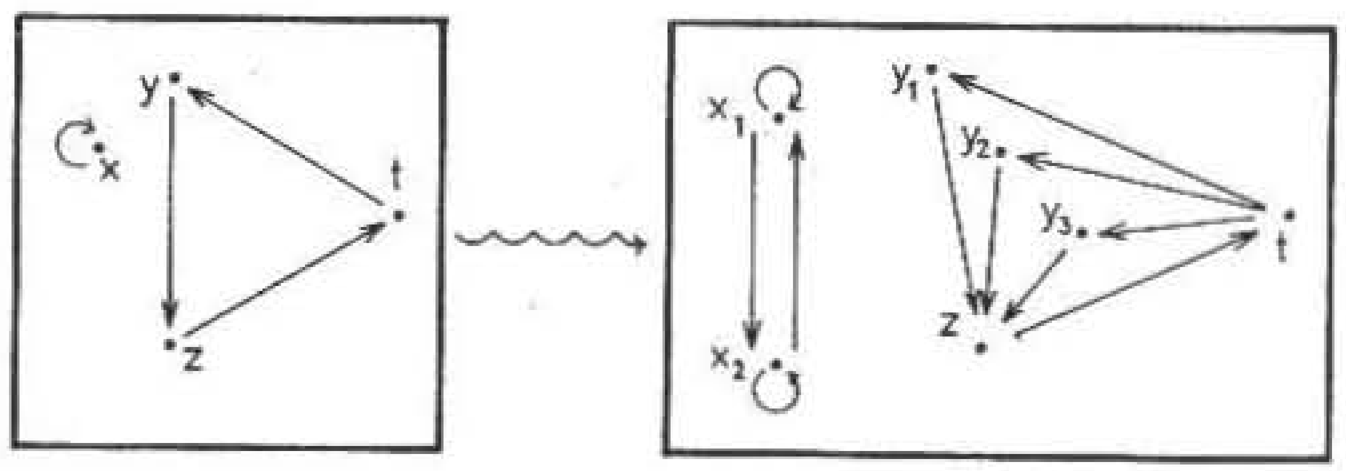
has an initial structure.

Both the initial structure α' and the structure α'' which makes Y a subobject are defined by the same condition: for each object (T,δ) and each map $h\colon T \to Y$, $v\,.\,h\colon (T,\delta) \to (X,\alpha)$ is a morphism iff $h\colon (T,\delta) \to (Y,\beta)$ is a morphism for $\beta = \alpha'$ or $\beta = \alpha''$.

Corollary. Each initially mono-complete construct is hereditary.

Conversely, the algebraic constructs are not initially mono-complete because they are not hereditary.

6. Splitting of points. In some constructs we can obtain new objects from the given ones by splitting their points (and their corresponding structure). For example, by splitting points x in a graph we obtain new points x_i, $i \in I(x)$; an arrow leads from x_i to y_j iff in the original graph an arrow leads from x to y. Example:



Or, in a topological space, we can split points and then consider the topology in which the open sets are precisely the sets $\hat{U}$, where U is open in the original topology and $\hat{U} = \{x_i;\ x \in U\}$. For example, by splitting the singleton space we obtain (all) indiscrete spaces.

Note that when splitting the points of a set X we obtain a set $\hat{X}$ together with a natural surjective map $f\colon \hat{X} \to X$ defined by $f(x_i) = x$. (Conversely, for each surjective map $f\colon \hat{X} \to X$ we can consider $\hat{X}$ as the result of a splitting of points of X: each point $x \in X$ is split into the points in $f^{-1}(x)$.) Both the split graph and the split topology are just the initial structures with respect to f – this can be easily derived from the examples in 2A1 and 2A2.

Definition. A *splitting* of an object (X,α) is an object $(\hat{X},\hat{\alpha})$ for which there exists a surjective morphism $f\colon \hat{X} \to X$ with $\hat{\alpha}$ the initial structure of the source $\{\hat{X} \xrightarrow{f} (X,\alpha)\}$.

A construct is said to *have splitting* if each singleton source $\{\hat{X} \xrightarrow{f} (X,\alpha)\}$ with f surjective has an initial structure.

Proposition. A construct is initially complete iff it has splitting and is initially mono-complete.

Proof. It is our task to show that for each initially mono-complete construct with splitting and each source

$$\{X \xrightarrow{f_i} (Y_i,\beta_i)\}_{i\in I}$$

an initial structure exists. (Its uniqueness then follows from the fact that an initially mono-complete construct is transportable, see Remark 2A3.)

Define an equivalence $\sim$ on X as follows: given $x, x' \in X$, then

$$x \sim x' \quad \text{iff} \quad f_i(x) = f_i(x') \qquad \text{for all} \quad i \in I\,.$$

Denote by $\varphi\colon X \to X/\!\sim$ the canonical map. For each $i \in I$ we can define $f_i'\colon X/\!\sim\ \to Y_i$ by

$$f_i'([x]) = f_i(x) \qquad \text{for all} \quad x \in X\,,$$

i.e., by

$$f_i'\,.\,\varphi = f_i \qquad (i \in I)\,.$$

The source

$$\{X/\!\sim\ \xrightarrow{f_i'} (Y_i,\beta_i)\}_{i\in I}$$

separates points: if $[x]$, $[x']$ are distinct equivalence-classes then $x \not\sim x'$, i.e., there exists $i \in I$ with $f_i(x) \neq f_i(x')$ or, in other words, $f_i'([x]) \neq f_i'([x'])$. Hence, this source has an initial structure α. Let $\hat{\alpha}$ be the initial structure (the splitting) of the singleton source $\{X \xrightarrow{\varphi} (X/\!\sim,\alpha)\}$. Then $\hat{\alpha}$ is initial with respect to the original source:

(1) All $f_i = f_i'\,.\,\varphi\colon (X,\hat{\alpha}) \to (Y_i,\beta_i)$ are morphisms;

(2) Given an object (T,δ) and a map $h\colon T \to X$ such that all

$$f_i\,.\,h = f_i'\,.\,(\varphi\,.\,h)\colon (T,\delta) \to (Y_i,\beta_i) \qquad (i \in I)$$

are morphisms then, necessarily,

$$\varphi\,.\,h\colon (T,\delta) \to (X/\!\sim,\alpha)$$

is a morphism. This, in turn, implies that

$$h\colon (T,\delta) \to (X,\hat{\alpha})$$

is a morphism. □

Examples. (i) The construct ***Pmet*** of pseudometric spaces. A *pseudometric* on a set X is a map $\alpha\colon X \to [0,+\infty)$ which fulfils the following two conditions:

$$\alpha(x,y) = \alpha(y,x) \qquad \text{for all} \quad x, y \in X\,;$$
$$\alpha(x,y) + \alpha(y,z) \geqq \alpha(x,z) \qquad \text{for all} \quad x, y, \in X\,.$$

(The case of metric is extended to allow $\alpha(x,y) = 0$ when $x \neq y$.)

The objects of ***Pmet*** are pseudometric spaces, i.e., pairs (X, α) where X is a set and α is a pseudometric. The morphisms from (X, α) to (Y, β) are contractions, i.e., maps $f\colon X \to Y$ such that

$$\beta(f(x), f(x')) \leqq \alpha(x, x') \qquad \text{for all} \quad x, x' \in X .$$

The construct ***Pmet*** has splitting. Let (X, α) be a pseudometric space. By splitting the points $x \in X$ we obtain new points x_i and we define a pseudometric $\hat{\alpha}$ by

$$\hat{\alpha}(x_i, y_j) = \alpha(x, y) .$$

More precisely, the initial pseudometric of a source $\{\hat{X} \xrightarrow{f} (X, \alpha)\}$ is defined by

$$\hat{\alpha}(x, x') = \alpha(f(x), f(x')) \qquad \text{for all} \quad x, x' \in X .$$

(ii) The construct $\boldsymbol{Pmet}_k$, $k \in (0, +\infty)$, of pseudometric spaces of diameter $\leqq k$. This is the full subconstruct of ***Pmet*** over spaces (X, α) with $\alpha(x, x') \leqq k$ for all $x, x' \in X$. This construct is initially complete. Indeed, $\boldsymbol{Pmet}_k$ has obviously splitting. And it is initially mono-complete – the proof is the same as for $\boldsymbol{Met}_k$ above.

7. Proposition (Initial structures are transitive.) Let

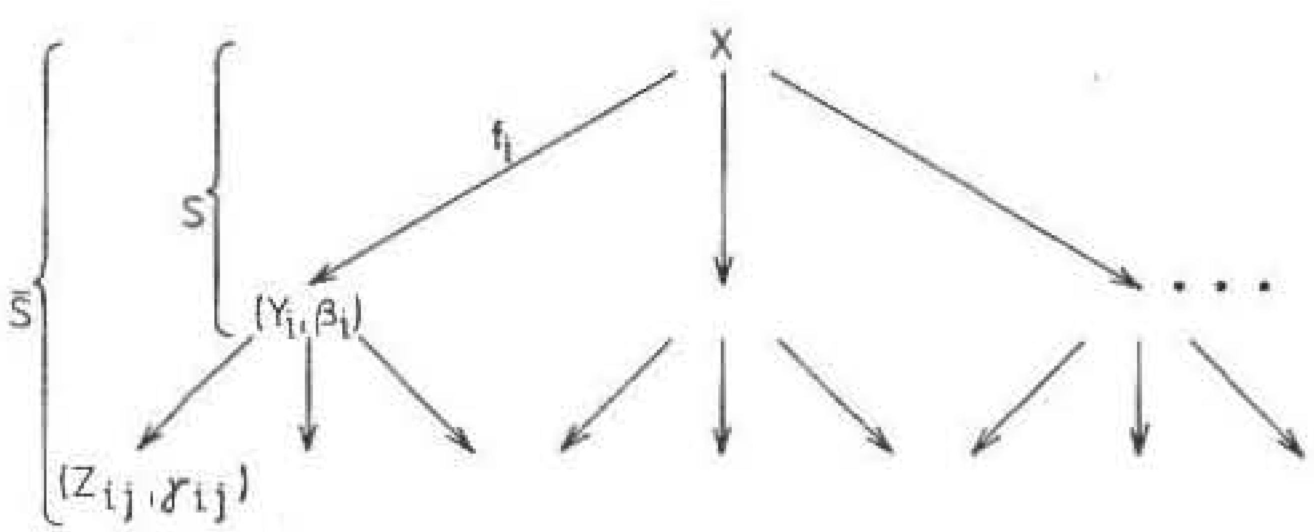
$$S = \{X \xrightarrow{f_i} (Y_i, \beta_i)\}_{i \in I}$$

be a source, and for each i let β_i be an initial structure of a source

$$S_i = \{Y_i \xrightarrow{g_{ij}} (Z_{ij}, \gamma_{ij})\}_{j \in J_i} .$$

Then a structure α on the set X is initial with respect to S iff it is initial with respect to the "composite" source

$$\bar{S} = \{X \xrightarrow{g_{ij} \cdot f_i} (Z_{ij}, \gamma_{ij})\}_{\substack{i \in I \\ j \in J_i}}$$



Proof. 1. Let α be initial with respect to S. Then each $f_i\colon (X, \alpha) \to (Y_i, \beta_i)$ is a morphism; hence, each $g_{ij} \cdot f_i\colon (X, \alpha) \to (Z_{ij}, \gamma_{ij})$ is a morphism. Further, if (T, δ) is an object and $f\colon T \to X$ is a mapping such that

$$g_{ij} \cdot f_i \cdot h\colon (T, \delta) \to (Z_{ij}, \gamma_{ij})$$

is a morphism for each $i \in I$ and $j \in J_i$, then

$$f_i \cdot h\colon (T, \delta) \to (Y_i, \beta_i)$$

is a morphism for each $i \in I$. Hence,

$$h\colon (T, \delta) \to (X, \alpha)$$

is a morphism.

2. Let α be initial with respect to $\bar{S}$. For each $i \in I$, all $g_{ij} \cdot f_i\colon (X, \alpha) \to (Z_{ij}, \gamma_{ij})$ are morphisms $(j \in J_i)$; hence, $f_i\colon (X, \alpha) \to (Y_i, \beta_i)$ is a morphism. Further, if (T, δ) is an object and $h\colon T \to X$ is a mapping such that all $f_i \cdot h\colon (T, \delta) \to (Y_i, \beta_i)$ are morphism then all

$$g_{ij} \cdot f_i \cdot h\colon (T, \delta) \to (Z_{ij}, \gamma_{ij})$$

are also morphisms. This implies that $h\colon (T, \delta) \to (X, \alpha)$ is a morphism. □

8. Concluding remark. By constructing the initial structures of sources, we obtain an important way of getting new objects from old. In some constructs this is always possible; in some it is possible for all sources separating points. And in the remaining constructs (notably, all those which are not hereditary) even simple sources can fail to have initial structures. Nevertheless, initial structures do appear even in these constructs for some important special sources. This will be seen in the following section.

Exercises 2A

a. Initial structures in a subconstruct. (1) Let $\mathcal{T}$ be a full subconstruct of a construct $\mathcal{S}$. Let $\{X \xrightarrow{f_i} (Y_i, \beta_i)\}$ be a source in $\mathcal{T}$. If α is the initial structure of this source in the construct $\mathcal{S}$ and if $\alpha \in \mathcal{T}[X]$, verify that α is initial in $\mathcal{T}$, too.

(2) Consider the lattice of 1Gc(2). The subposet $\{0, a_1, a_2, 1\}$ is evidently a lattice, too; nevertheless, it is not a sublattice (consider $a_1 \vee a_2$). Conclude that (1) does not hold for non-full subconstructs.

b. Initial algebraic structures.

(1) Prove that the following constructs of partial algebras:

$$\boldsymbol{Grd}_p, \boldsymbol{Mon}_p, \boldsymbol{Lat}_p$$

are initially mono-complete.

(2) Prove that the additive group of complex numbers $(\mathbb{K}, +, 0)$ is initial with respect to the source

$$\{\mathbb{K} \xrightarrow{p_j} (\mathbb{R}, +, 0)\}_{j = 1, 2} ,$$

where for each $x + iy \in \mathbb{K}$,

$$p_1(x + iy) = x \quad \text{and} \quad p_2(x + iy) = y .$$

(3) Prove the analogous statement about the ring of complex numbers (in ***Rng***). Why does the corresponding statement fail in ***Fld***?

(4) Prove that each finite-dimensional vector space is the initial object of a source of the following type:

$$\{X \xrightarrow{f_i} (\mathbb{R}, +, \cdot)\}_{i=1,\dots,n}.$$

c. Splitting of morphisms. Let $(\hat{X}, \hat{\alpha})$ be a splitting of an object (X, α) with respect to a surjective map $f\colon \hat{X} \to X$. Let $(\hat{Y}, \hat{\beta})$ be a splitting of (Y, β) with respect to $g\colon \hat{Y} \to Y$. By the splitting of a morphism $h\colon (X, \alpha) \to (Y, \beta)$ is meant an arbitrary map $\hat{h}\colon \hat{X} \to \hat{Y}$ such that $g \,.\, \hat{h} = h \,.\, f$.

(1) Prove that $\hat{h}\colon (\hat{X}, \hat{\alpha}) \to (\hat{Y}, \hat{\beta})$ is a morphism.

(2) Prove that each morphism in the construct ***Pros*** is a splitting of some morphism in ***Pos***.

d. A splitting cover of a construct $\mathscr{S}$ is a construct $\mathscr{S}^*$ such that (1) $\mathscr{S}^*$ has splitting and (2) $\mathscr{S}$ is a full subconstruct of $\mathscr{S}^*$ and (3) each object in $\mathscr{S}^*$ is a splitting of an object in $\mathscr{S}$, and each morphism in $\mathscr{S}^*$ is a splitting of a morphism in $\mathscr{S}$.

(1) Prove that ***Pros*** is a splitting cover of ***Pos***.

(2) Prove that ***Top*** is a splitting cover of ***Top***$_0$.

Hint: for each topological space (X, α) define the following equivalence $\sim$ on X:

$$x \sim x' \quad \text{iff} \quad x \in \overline{\{x'\}} \quad \text{and} \quad x' \in \overline{\{x\}}.$$

Then the quotient space is T_0 and (X, α) is its splitting.

(3) Prove that ***Pmet*** is a splitting cover of ***Met***.

(4) Prove that two splitting covers of a transportable construct must be concretely isomorphic.

(5) Prove that each transportable construct $\mathscr{S}$ has a splitting cover: the objects are $(X, \sim, \alpha)$, where X is a set, $\sim$ is an equivalence relation on X and $\alpha \in \mathscr{S}[X/\sim]$; the morphisms from $(X, \sim, \alpha)$ to $(Y, \approx, \beta)$ are the splittings of morphisms $h\colon (X/\sim, \alpha) \to (Y/\approx, \beta)$ in $\mathscr{S}$.

2B. Cartesian Products

1. Various structures on sets X_1 and X_2 are naturally transferred to the Cartesian product $X_1 \times X_2$. Let us illustrate this on the case of orderings.

Given posets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, define an ordering of the Cartesian product $X_1 \times X_2$ as follows:

$$(x_1, x_2) \leq (y_1, y_2) \quad \text{iff both} \quad x_1 \leq_1 y_1 \quad \text{and} \quad x_2 \leq_2 y_2.$$

This ordering is initial with respect to the source of projections,

$$\{X_1 \times X_2 \xrightarrow{\pi_i} (X_i, \leq_i)\}_{i=1,2}.$$

Recall that the projections are defined as follows:

$$\begin{aligned} \pi_1&\colon X_1 \times X_2 \to X_1; & \pi_1(x_1, x_2) &= x_1 \\ \pi_2&\colon X_1 \times X_2 \to X_2; & \pi_2(x_1, x_2) &= x_2 \end{aligned} \quad \text{for all} \quad (x_1, x_2) \in X_1 \times X_2.$$

Clearly,

$$\pi_1\colon (X_1 \times X_2, \leq) \to (X_1, \leq_1)$$

is order-preserving: for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$

$$(x_1, x_2) \leq (y_1, y_2) \quad \text{implies} \quad \pi_1(x_1, x_2) = x_1 \leq_1 y_1 = \pi_1(y_1, y_2).$$

Analogously, π_2 is order-preserving. Next, let $(T, \lesssim)$ be a poset and let $h\colon T \to X_1 \times X_2$ be a map such that

$$\pi_1 \,.\, h\colon (T, \lesssim) \to (X_1, \leq_1) \quad \text{and} \quad \pi_2 \,.\, h\colon (T, \lesssim) \to (X_2, \leq_2)$$

are order-preserving. Then

$$h\colon (T, \lesssim) \to (X_1 \times X_2, \leq)$$

is order-preserving. Given $t, t' \in T$ with $t \lesssim t'$ we put $h(t) = (x_1, x_2)$ and $h(t') = (x'_1, x'_2)$. Since $\pi_1 \,.\, h$ is order-preserving, we have

$$x_1 = \pi_1(x_1, x_2) = \pi_1 \,.\, h(t) \leq_1 \pi_1 \,.\, h(t') = \pi_1(x'_1, x'_2) = x'_1$$

and, similarly,

$$x_2 \leq_2 x'_2;$$

hence,

$$(x_1, x_2) \leq (x'_1, x'_2).$$

2. Definition. The *Cartesian product* of objects (X_1, α_1) and (X_2, α_2) is the object $(X_1 \times X_2, \alpha)$, where α is the initial structure of the following source

$$\{X_1 \times X_2 \xrightarrow{\pi_i} (X_i, \alpha_i)\}_{i=1,2}.$$

Examples.

(i) ***Top***: the Cartesian product of two topological spaces (X_1, α_1) and (X_2, α_2) is the space on $X_1 \times X_2$ with the following subbase

$$\alpha_0 = \{U_1 \times U_2;\, U_1 \in \alpha_1 \text{ and } U_2 \in \alpha_2\}.$$

In fact, by example 2A2(ii), a subbase for the topology on $X_1 \times X_2$ is

$$\alpha'_0 = \{U_1 \times X_2;\, U_1 \in \alpha_1\} \cup \{X_1 \times U_2;\, U_2 \in \alpha_2\}.$$

Now, $\alpha'_0 \subseteq \alpha_0$; and, on the other hand, each set in α_0 is the intersection of two sets in α'_0,

$$U_1 \times U_2 = (U_1 \times X_2) \cap (X_1 \times U_2).$$

Hence, α_0 and α'_0 are two subbases of the same topology.

For example, the product of two lines is the plane.

(ii) ***Met***: the Cartesian product of two metric spaces (X_1, α_1) and (X_2, α_2) is the set $X_1 \times X_2$ with the following metric:

$$\alpha((x_1, x_2); (y_1, y_2)) = \max\{\alpha_1(x_1, y_1); \alpha_2(x_2, y_2)\}.$$

It is easy to check that α is a metric. Furthermore:

(1) The projection $\pi_1: (X_1 \times X_2, \alpha) \to (X_1, \alpha_1)$ is a contraction, since

$$\alpha((x_1, x_2); (y_1, y_2)) \geqq \alpha_1(x_1, y_1) = \alpha_1(\pi_1(x_1, x_2); \pi_1(y_1, y_2)).$$

Similarly, π_2 is a contraction.

(2) Let (T, δ) be a metric space and let $h: T \to X_1 \times X_2$ be a map such that both $\pi_1 . h$ and $\pi_2 . h$ are contractions. Then h is also a contraction: given $t, t' \in T$ then

$$\alpha_1(\pi_1 . h(t); \pi_1 . h(t')) \leqq \delta(t, t'),$$

because $\pi_1 . h$ is a contraction, and

$$\alpha_2(\pi_2 . h(t); \pi_2 . h(t')) \leqq \delta(t, t'),$$

because $\pi_2 . h$ is a contraction. Thus,

$$\alpha(h(t), h(t')) = \max\{\alpha_1(\pi_1 . h(t), \pi_1 . h(t')); \alpha_2(\pi_2 . h(t); \pi_2 . h(t'))\} \leqq \delta(t, t').$$

For example, if $(X_1, \alpha_1) = (X_2, \alpha_2) = (\mathbb{R}, \varrho)$, the line with the Euclidean metric, then the product is the plane $\mathbb{R}^2$ with the following metric

$$\alpha(p, q) = \max\{|p_1 - q_1|; |p_2 - q_2|\} \qquad \text{for all} \quad p, q \in \mathbb{R}^2.$$

(iii) ***Grd***: the Cartesian product of two groupoids $(X_1, \circ)$ and $(X_2, *)$ is the groupoid $(X_1 \times X_2, \cdot)$, where

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \circ y_1, x_2 * y_2).$$

(1) The projection $\pi_1: (X_1 \times X_2, \cdot) \to (X_1, \circ)$ is a homomorphism, since for all (x_1, x_2) and (y_1, y_2) in $X_1 \times X_2$ we have

$$\begin{aligned}\pi_1((x_1, x_2) \cdot (y_1, y_2)) &= \pi_1(x_1 \circ y_1, x_2 * y_2)\\ &= x_1 \circ y_1\\ &= \pi_1((x_1, x_2)) \circ \pi_1((y_1, y_2)).\end{aligned}$$

Similarly, π_2 is a homomorphism.

(2) If $(T, +)$ is a groupoid and $h: T \to X_1 \times X_2$ is a map such that $\pi_1 . h$ and $\pi_2 . h$ are homomorphisms, then also h is a homomorphism: given $t, t' \in T$ put

$$h(t) = (x_1, x_2) \quad \text{and} \quad h(t') = (y_1, y_2).$$

Then

$$\pi_1 . h(t + t') = [\pi_1 . h(t)] \circ [\pi_1 . h(t')] = x_1 \circ y_1,$$

because $\pi_1 . h$ is a homomorphism; analogously,

$$\pi_2 . h(t + t') = x_2 * y_2.$$

Therefore,

$$\begin{aligned}h(t + t') &= (\pi_1 . h(t + t'), \pi_2 . h(t + t'))\\ &= (x_1 \circ y_1, x_2 * y_2)\\ &= (x_1, x_2) \cdot (y_1, y_2)\\ &= h(t) \cdot h(t').\end{aligned}$$

(iv) ***Sgr, Mon, Grp***: as in the case of groupoids, the operations are defined "coordinate-wise". Thus, given groups $(X_1, \circ, e_1)$ and $(X_2, *, e_2)$, their Cartesian product is the group $(X_1 \times X_2, \cdot, (e_1, e_2))$, where $\cdot$ is the operation of the preceding example.

3. More generally, we define the Cartesian product of a collection of objects. Recall that the Cartesian product of a family of sets $\{X_i; i \in I\}$ is the set

$$\prod_{i \in I} X_i$$

of all collections $x = \{x_i; i \in I\}$, where $x_i \in X_i$ for each $i \in I$. Thus,

$$X_1 \times X_2 = \prod_{i \in \{1,2\}} X_i, \qquad X_1 \times X_2 \times X_3 = \prod_{i \in \{1,2,3\}} X_i, \dots .$$

If $I = \mathbb{N}$ is the set of all natural numbers then

$$\prod_{i \in \mathbb{N}} X_i = X_0 \times X_1 \times X_2 \times X_3 \dots$$

is the set of all sequences the nth member of which is in X_n.

For each $i_0 \in I$ we have the i_0th *projection* from the Cartesian product $X = \prod_{i \in I} X_i$ into X_{i_0}, defined as follows

$$\pi_{i_0}: X \to X_{i_0}; \qquad \pi_{i_0}(x) = x_{i_0} \qquad \text{for each} \quad x \in X.$$

4. Definition. The *Cartesian product* of objects (X_i, α_i), $i \in I$, is the object (X, α), where

$$X = \prod_{i \in I} X_i$$

and α is an initial structure of the source of projections

$$\{X \xrightarrow{\pi_i} (X_i, \alpha_i)\}_{i \in I}.$$

Remarks. (i) If each collection of objects has a Cartesian product, we say that the construct *has Cartesian products*.

(ii) The Cartesian product of objects A_i, $i \in I$, is also denoted by $\prod_{i \in I} A_i$.

Examples. (i) ***Pos***: given posets $(X_i, \leqq_i)$, $i \in I$, their Cartesian product is the poset $(\prod X_i, \leqq)$, where

$$x \leqq y \quad \text{iff} \quad x_i \leqq_i y_i \quad \text{for each} \quad i \in I \qquad (x, y \in \prod X_i).$$

(ii) ***Grd***: given groupoids $(X_i, \circ_i)$, $i \in I$, their Cartesian product is the groupoid $(\prod X_i, \cdot)$, where

$$x \cdot y = \{x_i \circ_i y_i\}_{i \in I} \quad (x, y \in \prod X_i).$$

(iii) ***Top*** and ***Topc***: the Cartesian product of topological spaces (X_i, α_i), $i \in I$, is the space on $X = \prod X_i$, the subbase of which is

$$\alpha_0 = \{\pi_i^{-1}(U);\ i \in I \text{ and } U \in \alpha_i\}.$$

It is a non-trivial topological theorem that ***Topc*** has Cartesian products.

Tychonoff theorem: the Cartesian product of compact spaces is compact. Clearly, the product of T_0 spaces is T_0; analogously with T_1, T_2. Hence, the constructs

$$\mathbf{Top}_0,\ \mathbf{Top}_1,\ \mathbf{Top}_2,\ \mathbf{Topc} \text{ and } \mathbf{Comp}$$

have Cartesian products (see 2Aa).

Observation. Each initially mono-complete construct has Cartesian products. Indeed, the source of projections

$$\{X \xrightarrow{\pi_i} X_i\}_{i \in I}, \quad \text{where} \quad X = \prod X_i,$$

separates points: if $x = \{x_i\}$ and $y = \{y_i\}$ are distinct then there exists $i \in I$ with $x_i \neq y_i$ – hence,

$$\pi_i(x) \neq \pi_i(y).$$

Thus,

$$\mathbf{Gra},\ \mathbf{Top},\ \mathbf{Pos},\ \mathbf{Met}_k$$

have Cartesian products.

On the other hand, algebraic constructs have Cartesian products though they are not initially mono-complete; the operations are defined coordinate-wise. We have seen this in ***Grd***; analogously with

$$\mathbf{Sgr},\ \mathbf{Mon},\ \mathbf{Grp},\ \mathbf{Rng}.$$

For example, given rings

$$(X_i, +_i, 0_i, \cdot_i, e_i) \qquad i \in I,$$

their Cartesian product is the ring

$$(X, +, 0, \cdot, e),$$

where $X = \prod_{i \in I} X_i$ and

$$\left.\begin{aligned} (x + y)_i &= x_i +_i y_i \\ (x \cdot y)_i &= x_i \cdot_i y_i \end{aligned}\right\} \quad \text{for all } x, y \in X \text{ and } i \in I;$$

$$0 = \{0_i\}; \quad e = \{e_i\}.$$

Let us mention some examples where Cartesian products fail.

Examples. (iv) ***Met***: let (X_i, α_i), $i \in I$, be metric spaces; for each $x, y \in X = \prod X_i$ put

$$\alpha(x, y) = \bigvee_{i \in I} \alpha_i(x_i, y_i).$$

Then $\alpha(x, y)$ is either a real number or ∞. If $\alpha(x, y)$ is real for all $x, y \in X$, then α is a metric, and (X, α) is the Cartesian product. This is proved as in 2B2(ii). If there exist $x^0, y^0 \in X$ with $\alpha(x^0, y^0) = \infty$ then the collection of metric spaces fails to have a Cartesian product. Let β be a metric such that (X, β) is the Cartesian product; then π_i is a contraction, hence

$$\beta(x^0, y^0) \geqq \alpha_i(x_i^0, y_i^0) \qquad \text{for each} \quad i \in I.$$

This contradicts to

$$\bigvee_{i \in I} \alpha_i(x_i^0, y_i^0) = \alpha(x^0, y^0) = \infty.$$

(v) ***Fld***: no two non-trivial fields have a Cartesian product. Indeed, all morphisms in ***Fld*** are one-to-one or constant, but the projections are neither.

Note that, for two non-trivial fields $(X_i, +_i, \cdot_i, 0_i, e_i)$, $i = 1, 2$, the product in ***Rng***,

$$(X_1 \times X_2, +, \cdot, (0_1, 0_2), (e_1, e_2))$$

is not a field because the elements $(0_1, x_2)$, $x_2 \in X_2$, fail to have a (multiplicative) inverse.

5. Theorem. A fibre-small construct is initially mono-complete iff it has Cartesian products and is hereditary.

Proof. Each initially mono-complete construct is hereditary (2A5) and has products (by the preceding observation). Conversely, let $\mathscr{S}$ be a fibre-small, hereditary construct with Cartesian products. For each source

$$\{X \xrightarrow{f_i} (X_i, \alpha_i)\}_{i \in I}$$

which separates points we shall find an initial structure.

I. First, suppose I is a set (not a large class). Then we can form the Cartesian product (Y, β) of the objects (X_i, α_i), $i \in I$, and we define a map

$$f\colon X \to Y = \prod X_i$$

by

$$f(x) = \{f_i(x)\} \qquad \text{for each} \quad x \in X,$$

i.e., by $f_i = \pi_i . f$ $(i \in I)$. Since the source separates points, f is clearly one-to-one. Put

$$X' = f(X) \subseteq Y$$

and denote by

$$f'\colon X \to X'$$

the bijection which is a restriction of f. Then

$$f = v . f',$$

where $v\colon X' \to Y$ is the inclusion map. Since the construct $\mathscr{S}$ is hereditary, a structure α' on X' exists such that (X', α') is a subobject of (Y, β); since $\mathscr{S}$ is fibre-small (hence, transportable), there exists a structure α on X such that $f\colon (X, \alpha) \to (X', \alpha')$ is an isomorphism. Let us check that α is initial.

(1) Each $f_i\colon (X, \alpha) \to (X_i, \alpha_i)$, $i \in I$, is a morphism since it is composed of three morphisms:

$$f_i = \pi_i . f = \pi_i . v . f' \qquad \text{for each} \quad i \in I .$$

(2) Let (T, δ) be an object and let $h\colon T \to X$ be a map such that each

$$f_i . h = \pi_i . (v . f' . h)\colon (T, \delta) \to (X_i, \alpha_i), \qquad i \in I,$$

is a morphism. By the definition of Cartesian product,

$$v . f' . h\colon (T, \delta) \to (Y, \beta)$$

is a morphism. By the definition of subobject,

$$f' . h\colon (T, \delta) \to (X', \alpha')$$

is a morphism. Hence,

$$h = (f')^{-1} . (f' . h)\colon (T, \delta) \to (X, \alpha)$$

is a morphism.

II. If I is a proper class, we shall find a subset $I_0 \subseteq I$ such that the restricted source has the same initial structure as the original one. Note that all equivalences on the set X form a set. Since the construct $\mathscr{S}$ is fibre-small, all objects $(X/\!\sim, \gamma)$, where $\sim$ is an arbitrary equivalence, also form a set.

For each $i \in I$ let $\sim_i$ be the kernel equivalence of f_i; then f_i can be factored as

$$f_i = v_i . \bar{f}_i . \varphi_i ,$$

where

$$\varphi_i\colon X \to X/\!\sim_i$$

is the canonical morphism,

$$\bar{f}_i\colon X/\!\sim_i \to f_i(X)$$

is a bijection, and

$$v_i\colon f_i(X) \to X_i$$

is the inclusion map. Let $(f_i(X), \alpha_i')$ be a subobject of (X_i, α_i) (recall that $\mathscr{S}$ is hereditary), and let γ_i be a structure on $X/\!\sim_i$ such that

$$\bar{f}_i\colon (X/\!\sim_i, \gamma_i) \to (f_i(X), \alpha_i')$$

is an isomorphism. By the fibre-smallness of $\mathscr{S}$, there exists a subset $I_0 \subseteq I$ such that

(*) for each $i \in I$ we can find $j \in I_0$ with $\sim_i$ equal to $\sim_j$ and γ_i equal to γ_j.

The restricted source

$$\{X \xrightarrow{f_j} (X_j, \alpha_j)\}_{j \in I_0}$$

separates points: Given distinct $x, x' \in X$ there exists $i \in I$ with $f_i(x) \neq f_i(x')$, i.e., with $x \not\sim_i x'$; find $j \in I_0$ as in (*), then $f_j(x) \neq f_j(x')$. By I., this restricted source has an initial structure α.

To prove that α is initial with respect to the original source, it clearly suffices to prove that $f_i\colon (X, \alpha) \to (X_i, \alpha_i)$ is a morphism for each $i \in I$. Find j as in (*). Since $f_j\colon (X, \alpha) \to (X_j, \alpha_j)$ is a morphism and $f_j = v_j . \bar{f}_j . \varphi_j$, clearly

$$\bar{f}_j . \varphi_j\colon (X, \alpha) \to (f_j(X), \alpha_j')$$

is also a morphism; hence,

$$\varphi_j = \bar{f}_j^{-1} . (\bar{f}_j . \varphi_j)\colon (X, \alpha) \to (X/\!\sim_j, \gamma_j)$$

is a morphism. In other words,

$$\varphi_i\colon (X, \alpha) \to (X/\!\sim_i, \gamma_i)$$

is a morphism. This implies that

$$f_i = \varphi_i . \bar{f}_i . v_i = (X, \alpha) \to (X_i, \alpha_i)$$

is a morphism. □

6. **Observation.** The Cartesian product $A = \prod_{i \in I} A_i$ has the following *universal property*: the projections form a collection of morphisms

$$\pi_i\colon A \to A_i, \qquad i \in I,$$

such that for each collection of morphisms

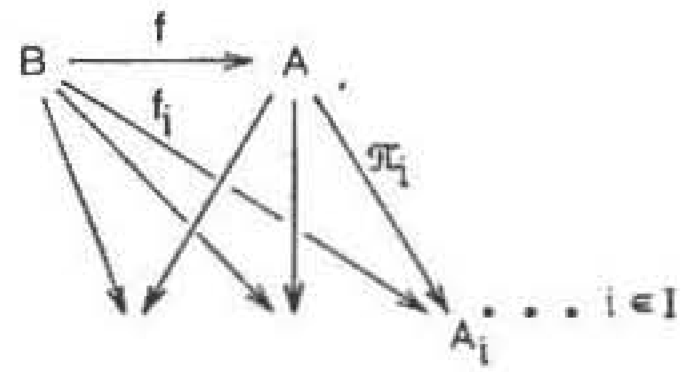
$$f_i\colon B \to A_i, \qquad i \in I,$$

there exists a unique morphism

$$f\colon B \to A$$

with

$$f_i = \pi_i . f \qquad \text{for each} \quad i \in I .$$



Indeed, if $A_i = (X_i, \alpha_i)$ and $B = (Y, \beta)$ then f is defined by

$$f(y) = \{f_i(y)\}_{i \in I} \qquad \text{for each} \quad y \in Y .$$

This is the unique map $f\colon Y \to \prod X_i$ with $f_i = \pi_i . f$ $(i \in I)$. And f is a morphism because $\pi_i . f = f_i\colon B \to A_i$ are morphisms for all $i \in I$.

7. We conclude this section by a proposition which shows that a lot of constructs, though not hereditary, admit the formation of subobjects on all sets defined by the "coincidence" of two morphisms.

Definition. Let

$$f, g\colon (X, \alpha) \to (Y, \beta)$$

be two morphisms with common domain and common range. By the *equalizer* of f and g we mean a subobject of (X, α) on the set

$$E = \{x \in X; f(x) = g(x)\} .$$

Remark. Let (E, α') be the equalizer, and let

$$v\colon (E, \alpha') \to (X, \alpha)$$

be the inclusion morphism. This morphism has the following universal property:

(i) $f . v = g . v$;

(ii) for each morphism $h\colon (T, \delta) \to (X, \alpha)$ with

$$f . h = g . h ,$$

there exists a unique morphism $h'\colon (T, \delta) \to (E, \alpha')$ such that

$$h = v . h' .$$

$$\begin{tikzcd}
(E,\alpha') \arrow[r, "v"] & (X,\alpha) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,\beta) \\
& (T,\delta) \arrow[ul, "h'"] \arrow[u, "h"'] &
\end{tikzcd}$$

Proposition. Let $\mathscr{S}$ be a transportable construct with intersections (1F4) and with Cartesian products of pairs of objects. Then for arbitrary morphisms

$$f, g\colon (X, \alpha) \to (Y, \beta)$$

the equalizer exists.

Proof. Let $(X \times Y, \gamma)$ be the product of (X, α) and (Y, β); the projections will be denoted by π_X and π_Y.

I. The subset

$$M_f = \{(x, f(x)); x \in X\} \subseteq X \times Y$$

is a subobject of $(X \times Y, \gamma)$. Indeed, define a morphism

$$\bar{f}\colon (X, \alpha) \to (X, \alpha) \times (Y, \beta)$$

by $\bar{f}(x) = (x, f(x))$ for all $x \in X$; i.e., by

$$\pi_X . \bar{f} = \mathrm{id}_X \quad \text{and} \quad \pi_Y . \bar{f} = f .$$

(Since id_X and f are morphisms, so is $\bar{f}$.) Since $\bar{f}$ is clearly one-to-one and

$$\bar{f}(X) = M_f ,$$

we have

$$\bar{f} = v . \bar{f}_0 ,$$

where $v\colon M_f \to X \times Y$ is the inclusion map and $\bar{f}_0$ is a bijection. Denote by $\bar{\alpha}$ the structure transported by $\bar{f}_0$, i.e., such that $\bar{f}_0\colon (X, \alpha) \to (M_f, \bar{\alpha})$ is an isomorphism. We shall verify that $(M_f, \bar{\alpha})$ is a subobject of $(X \times Y, \gamma)$.

$$\begin{tikzcd}
(X,\alpha) \arrow[rr, shift left, "\bar{f}"] \arrow[drr, "\bar{f}_0"] & & (X \times Y,\gamma) \arrow[ll, shift left, "\pi_X"] \\
& & (M_f,\bar{\alpha}) \arrow[u, "v"'] \\
(T,\delta) \arrow[urr, "h"] & &
\end{tikzcd}$$

(1) Since $\bar{f} = v . \bar{f}_0$ and since $\bar{f}$ and $\bar{f}_0^{-1}$ are morphisms,

$$v = \bar{f} . \bar{f}_0^{-1}\colon (M_f, \alpha) \to (X \times Y, \gamma)$$

is also a morphism.

(2) Note that

$$\bar{f}_0 . \pi_X . v = \mathrm{id}_{M_f}$$

[because for each $(x, f(x)) \in M_f$ we have $\bar{f}_0 . \pi_X . v(x, f(x)) = \bar{f}_0(x) = (x, f(x))$]. Let (T, δ) be an object and let $h: T \to M_f$ be a map such that

$$v . h: (T, \delta) \to (X \times Y, \gamma)$$

is a morphism. Then

$$h = \bar{f}_0 . \pi_X . v . h: (T, \delta) \to (M_f, \bar{\alpha})$$

is also a morphism since it is composed of three morphisms: $\bar{f}_0$, π_X and $v . h$.

II. The subset

$$M_g = \{(x, g(x)); x \in X\} \subseteqq X \times Y$$

is a subobject – the proof is analogous. Hence, the intersection

$$M_f \cap M_g = \{(x, f(x)); x \in E\}$$

is a subobject, too. Denote by γ_0 the corresponding structure and by

$$v_0: (M_f \cap M_g, \gamma_0) \to (X \times Y, \gamma)$$

the inclusion morphism.

III. We can restrict $\bar{f}$ to a bijection

$$\bar{f}: E \to M_f \cap M_g .$$

Let α_0 be the structure transported by $\bar{f}$, i.e., such that

$$\bar{f}: (E, \alpha_0) \to (M_f \cap M_g, \gamma_0)$$

is an isomorphism. We shall verify that (E, α_0) is a subobject of (X, α). Denote by $w: E \to X$ the inclusion map.

$$\begin{tikzcd}
(T,\delta) \arrow[r, "h"] & (E,\alpha_0) \arrow[r, "\bar{f}"] \arrow[d, "w"'] & (M_f \cap M_g, \gamma_0) \arrow[d, "v_0"] \\
 & (X,\alpha) \arrow[dr, "f"] & (X \times Y, \gamma) \arrow[l, "\pi_X"'] \arrow[d, "\pi_Y"] \\
 & & (Y,\beta)
\end{tikzcd}$$

First, observe that

$$w = \pi_X . v_0 . \bar{f}: E \to X$$

and

$$\pi_Y . v_0 = f . \pi_X . v_0: M_f \cap M_g \to X .$$

(1) The first of these equalities implies that

$$w: (E, \alpha_0) \to (X, \alpha)$$

is a morphism.

(2) Let (T, δ) be an object and let $h: T \to E$ be a map such that $w . h: (T, \delta) \to (X, \alpha)$ is a morphism. Then both

$$\pi_X . (v_0 . \bar{f} . h) = w . h: (T, \delta) \to (X, \alpha)$$

and

$$\pi_Y . (v_0 . \bar{f} . h) = f . \pi_X . v_0 . \bar{f} . h = f . (w . h): (T, \delta) \to (Y, \beta)$$

are morphisms. This proves that

$$v_0 . \bar{f} . h: (T, \delta) \to (X \times Y, \gamma)$$

is a morphism. Since v_0 is the inclusion of a subobject, $\bar{f} . h: (T, \delta) \to (M_f \cap M_g, \gamma_0)$ is also a morphism. Hence,

$$h = \bar{f}^{-1} . (\bar{f} . h): (T, \delta) \to (E, \alpha_0)$$

is a morphism. □

Examples. (i) ***Lat*** has equalizers. Let $f, g: (X, \leqq) \to (Y, \leqq)$ be lattice homomorphisms. Then E is a sublattice of $(X, \leqq)$: Given $x_1, x_2 \in E$ then

$$f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = g(x_1) \vee g(x_2) = g(x_1 \vee x_2),$$

which means that $x_1 \vee x_2 \in E$; analogously, $x_1 \wedge x_2 \in E$.

Analogously with other algebraic constructs (***Grp***, ***Vect***, ***Rng***).

(ii) ***Comp*** has equalizers. Let $f, g: (X, \alpha) \to (Y, \beta)$ be continuous maps in ***Comp***; then E is a subobject, i.e., a closed subset of (X, α). Given $x \in \bar{E}$ then $f(x) = g(x)$ (i.e., $x \in E$): If not, choose disjoint open sets U, containing $f(x)$, and V, containing $g(x)$. Since $f^{-1}(U) \cap g^{-1}(V)$ is an open set, containing x but disjoint from E, this is a contradiction.

Exercises 2B

a. Products

(1) ***Lat***: prove that the Cartesian product $\prod(X_i, \leqq_i)$ of posets is a lattice whenever each $(X_i, \leqq_i)$ is a lattice; the joins and meets are formed coordinate-wise. Conclude that ***Lat*** has Cartesian products (2Aa).

Does the same hold for ***Csl*** and ***Clat***?

(2) ***Vect***: verify that the Cartesian product of vector spaces is a vector space with "coordinate-wise" operations.

(3) ***Nor***: verify that two normed vector spaces have a Cartesian product but that no infinite collection of non-trivial normed vector spaces has a Cartesian product. Hint: see Example 2B4(iv); note that since $|r \cdot x| = |r| \cdot |x|$, a norm is always unbounded.

b. Equalizers: (i) ***Grd***: verify explicitly that for arbitrary homomorphisms $f, g\colon (X, \circ) \to (Y, \cdot)$ the set E is a subgroupoid of $(X, \circ)$. Does the same hold in ***Sgr*** and ***Grp***?

(ii) ***Clat***: check the equalizers for complete lattice homomorphisms.

(iii) ***Topc***: find two morphisms which fail to have an equalizer. Hint: see Remark 1F4; define $f, g\colon X \to X = \{\infty_1, \infty_2\} \cup \{0, 1, 2, \ldots\}$ by $f(x) = g(x)$ for all $x \in \{0, 1, 2, \ldots\}$, $f(\infty_1) = \infty_1 = g(\infty_2)$ and $g(\infty_1) = \infty_2 = f(\infty_2)$.

2C. Final Structures

1. Final structures are defined "dually" to the initial structures: the arrows lead from objects to a set.

A *sink* in a construct $\mathscr{S}$ on a set X is a (possibly large) collection (Y_i, β_i, f_i), $i \in I$, where (Y_i, β_i) are objects of $\mathscr{S}$ and $f_i\colon Y_i \to X$ are maps. The following notation is used:

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}.$$

Definition. A *final structure* of a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$ is a structure α on X such that

(1) $f_i\colon (Y_i, \beta_i) \to (X, \alpha)$ are morphisms for all $i \in I$;

(2) for each object (T, δ) and each map $h\colon X \to T$ such that all $h \,.\, f_i\colon (Y_i, \beta_i) \to (T, \delta)$ are morphisms $(i \in I)$, also $h\colon (X, \alpha) \to (T, \delta)$ is a morphism.

$$\begin{tikzcd} (Y_i,\beta_i) \arrow[d, "f_i"'] & \cdots \arrow[dl] \\ (X,\alpha) \arrow[d, "h"] & \\ (T,\delta) & \end{tikzcd}$$

Examples. (i) ***Gra***: the final structure of a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$ is the following graph α on X:

$$\alpha = \{(x, x') \in X \times X;\ (x, x') = (f_i(y), f_i(y')) \text{ for some } i \in I \text{ and } (y, y') \in \beta_i\}.$$

Proof.

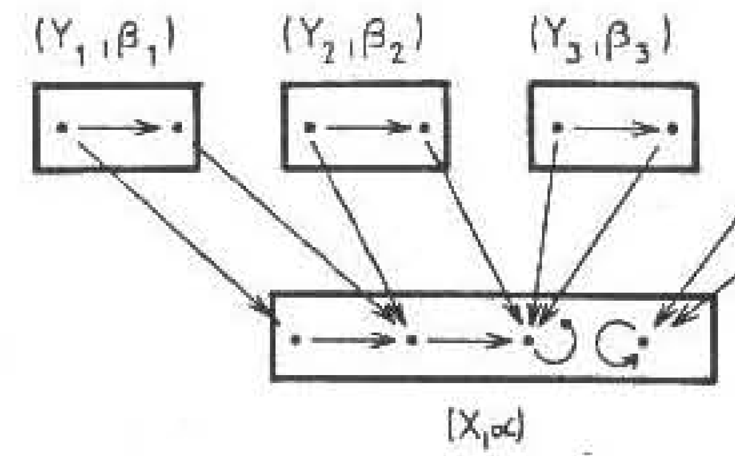
(1) $f_i\colon (Y_i, \beta_i) \to (X, \alpha)$ are compatible since $(y, y') \in \beta_i$ implies $(f_i(y), f_i(y')) \in \alpha$.

(2) If (T, δ) is a graph and $h\colon X \to T$ is a map such that all $h \,.\, f_i\colon (Y_i, \beta_i) \to (T, \delta)$ are compatible then $h\colon (X, \alpha) \to (T, \delta)$ is also compatible. For each $(x, x') \in \alpha$ we have $(x, x') = (f_i(y), f_i(y'))$, where $i \in I$ and $(y, y') \in \beta_i$; then

$$(h(x), h(x')) = (h \,.\, f_i(y), h \,.\, f_i(y')) \in \delta$$

since $h \,.\, f_i$ is compatible.

A concrete example:



(ii) ***Top***: the final structure of a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}$ is the following topology α:

$$U \in \alpha \quad \text{iff} \quad f_i^{-1}(U) \in \beta_i \qquad \text{for all} \quad i \in I.$$

Proof: (1) $f_i\colon (Y_i, \beta_i) \to (X, \alpha)$ are continuous, since $U \in \alpha$ implies $f_i^{-1}(U) \in \beta_i$;

(2) If (T, δ) is a topological space and $h\colon X \to T$ is a map with each $h \,.\, f_i$ continuous then f is also continuous. For each $V \in \delta$ we have

$$f_i^{-1} \,.\, (h^{-1}(V)) = (h \,.\, f_i)^{-1}(V) \in \beta_i \qquad \text{for all} \quad i \in I.$$

Thus, $h^{-1}(V) \in \alpha$.

2. Remark. A construct is said to be *finally complete* if each sink has a unique final structure. As in 2A3 it can be shown that a finally complete construct is transportable. We are going to prove that initial and final completeness are equivalent properties.

For each sink

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$$

consider the (large) collection of all objects (T, δ) and all maps $h\colon X \to T$ such that

$$\text{each} \quad h \,.\, f_i\colon (Y_i, \beta_i) \to (T, \delta) \quad \text{is a morphism} \quad (i \in I).$$

This collection can be written with the use of indices, say, as (T_j, δ_j, h_j), $j \in J$

(where J is an auxiliary index class; we assume that the triples (T_j, δ_j, h_j) are pairwise distinct). The source

$$\{X \xrightarrow{h_j} (T_j, \delta_j)\}_{j \in J}$$

is called the *dual source* of the given sink.

$$\begin{tikzcd} (Y_i,\beta_i) \arrow[d, "f_i"'] & \cdots \arrow[dl] & i \in I \\ X \arrow[d, "h_j"'] \arrow[dr] & & \\ (T_j,\delta_j) & \cdots & j \in J \end{tikzcd}$$

Thus, the dual source of a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i\in I}$ is a source $\{X \xrightarrow{h_j} (T_j, \delta_j)\}_{j\in J}$ which is maximal with respect to the following properties:

(i) each $h_j . f_i$: $(Y_i, \beta_i) \to (T_j, \delta_j)$ is a morphism ($i \in I$ and $j \in J$);

(ii) the triples (T_j, δ_j, h_j) are pairwise distinct.

Analogously, the *dual sink* of a source $\{X \xrightarrow{h_j} (T_j, \delta_j)\}_{j\in J}$ is a sink $\{(Y_i, \beta_i \xrightarrow{f_i} X\}_{i\in I}$ which is maximal with respect to (i) and

(ii') the triples (Y_i, β_i, f_i) are pairwise distinct.

Duality Theorem. A construct is initially complete iff it is finally complete.

Proof. Let $\mathscr{S}$ be initially complete, and let

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i\in I}$$

be its sink. We prove that the initial structure α of the dual source

$$\{X \xrightarrow{h_j} (T_j, \delta_j)\}_{j\in J}$$

is final for the given sink.

(1) For each $i \in I$,

$$f_i: (Y_i, \beta_i) \to (X, \alpha)$$

is a morphism. This follows from the initiality of α since all $h_j . f_i$: $(Y_i, \beta_i) \to (T_j, \delta_j)$ are morphisms ($j \in J$).

(2) Let (T, δ) be an object and h: $X \to T$ a map such that all $h . f_i$: $(Y_i, \beta_i) \to$ $\to (T, \delta)$ are morphisms ($i \in I$). By the maximality of the dual source, there exists $j \in J$ with $(T, \delta, h) = (T_j, \delta_j, h_j)$. This implies that h: $(X, \alpha) \to (T, \delta)$ is a morphism.

The uniqueness of the final structure α follows from the fact that $\mathscr{S}$ is transportable (2A3): if α' is another final structure then, obviously, α and α' are equivalent and hence equal.

Conversely let $\mathscr{S}$ be finally complete. Then each source has an initial structure: this is the final structure of the dual sink. And the uniqueness follows, again, from the fact that $\mathscr{S}$ is transportable. □

Example: the construct ***Pros*** is initially and hence also finally complete. For each sink of preordered sets

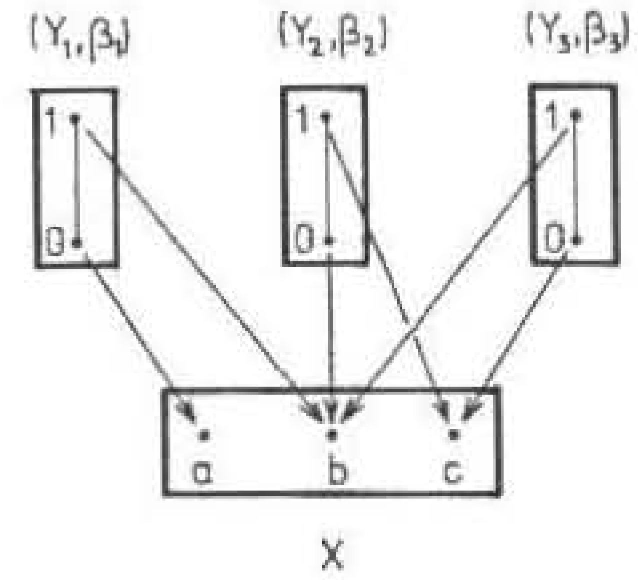
$$\{(Y_i, \leqq_i) \xrightarrow{f_i} X\}_{i\in I},$$

let α be the final graph on X (2Cl(i)). Let $\leqq$ be the smallest preorder, containing α, i.e., for each $x, x' \in X$,

$$(*) \qquad x \leqq x' \quad \text{iff} \quad x = x' \quad \text{or there are} \quad x = t_0, t_1, \ldots, t_n = x' \quad \text{in } X$$
$$\text{with } t_0 \,\alpha\, t_1,\ t_1 \,\alpha\, t_2,\ \ldots,\ t_{n-1} \,\alpha\, t_n .$$

Then $\leqq$ is the final preorder. Clearly, each f_i is order-preserving: $y \leqq_i y'$ implies $f_i(y) \,\alpha\, f_i(y')$, hence, $f_i(y) \leqq f_i(y')$ (for all $i \in I$; $y, y' \in Y$). If $(T, \precsim)$ is a preordered set and h: $X \to Y$ is a map with $h . f_i$ order-preserving for all $i \in I$, then h: $(X, \alpha) \to$ $\to (T, \precsim)$ is compatible; hence, h: $(X, \leqq) \to (T, \precsim)$ is order-preserving – see (*).

Consider the following sink, where each $(Y_i, \leqq_i)$ is the poset $(\{0, 1\}, \leqq)$ with $0 \leqq 1$ and $1 \nleqq 0$:



Then $\leqq$ is the following relation:

$$a \leqq b \quad \text{and} \quad b \leqq c; \qquad c \leqq b$$

(while $b \nleqq a$).

Remark. The formation of quotient objects is a special case of final structures. For each object (X, α) and each equivalence $\sim$ on X consider the singleton sink

$$\{(X, \alpha) \xrightarrow{\varphi} X/\!\sim\}.$$

Then an object $(X/\!\sim, \bar{\alpha})$ is final iff it is the quotient object of (X, α) under the equivalence $\sim$; this is similar to the situation with subobjects (2A5).

Consequently, every initially complete construct is cohereditary.

3. A sink on a set X can also be empty: the indexing class I is the empty set (so that no object (Y_i, β_i) and no map f_i are actually given). A structure α is final with respect to the empty sink iff for each object (T, δ) and each map h: $X \to T$,

$$h: (X, \alpha) \to (T, \delta) \quad \text{is a morphism}.$$

This is precisely the definition of a discrete object (1H4).

Analogously, we can define an *indiscrete object* on a set X as an object (X, α) such that for each object (T, δ) every map h: $T \to X$ is a morphism, h: $(T, \delta) \to (X, \alpha)$. Equivalently: α is the initial structure of the empty source on X. (This terminology is consistent with that for topological spaces (1C6).)

Observation. An initially complete construct has a discrete and an indiscrete object on each set.

Example: in ***Pmet***$_k$, the discrete pseudometric on X is defined by

$$\alpha(x, y) = \begin{cases} k & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \qquad \text{for all } x, y \in X;$$

the indiscrete one by $\alpha(x, y) = 0$ $(x, y \in X)$.

Exercises 2C

a. Final order. For which sinks in ***Pos*** does a final order exist? Hint: inspect the final preorder of the sink.

b. Final pseudometric. (1) Let $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}$ be a sink in ***Pmet***$_k$. Given x, x' in X, denote by $\alpha(x, x')$ the infimum (in $[0, k]$) of all the sums $\sum_{m=0}^{n} \beta_{i_m}(y_m, y'_m)$ where $i_0, \dots, i_n \in I$ and $y_m, y'_m \in Y_{i_m}$ fulfil the following condition:

$$x = f_{i_0}(y_0), \quad x' = f_{i_n}(y'_n) \quad \text{and} \quad f_{i_m}(y'_m) = f_{i_{m+1}}(y_{m+1}).$$

Prove that α is the final pseudometric of the given source.

(2) Exhibit a sink in ***Pmet*** which has no final structure.

(3) Exhibit a sink in ***Met***$_k$ which has no final structure.

c. The transitivity of final structures. Formulate and prove the statement analogous to 2A7.

d. Disjoint union. Let (X_i, α_i), $i \in I$, be objects with the sets X_i pairwise disjoint; put $X = \bigcup_{i \in I} X_i$. Then for each $i \in I$ we have the inclusion map $v_i\colon X_i \to X$. The disjoint union of the given objects is the final object of the following sink:

$$\{(X_i, \alpha_i) \xrightarrow{v_i} X\}_{i \in I}.$$

(1) Describe disjoint unions in ***Pos***, ***Top*** and ***Met***$_k$. Hint: in ***Met***$_k$ the distance of $x \in X_i$ and $y \in X_j$ is k whenever $i \neq j$.

(2) Show that disjoint unions generally do not exist in ***Lat***, ***Comp*** and ***Met***.

(3) Verify that unary algebras have disjoint unions but that other algebraic constructs, e.g., ***Grp*** and ***Vect***, do not.

2D. Semifinal Objects*

1. While initial completeness (or final completeness) is a rather special property of constructs, we present a generalization which is encountered in a large number of current constructs: semifinal completeness. To explain the idea, we start with sinks in ***Pos***.

For each sink of posets

$$\{(Y_i, \leq_i) \xrightarrow{f_i} X\}_{i \in I}$$

we have the final preorder $\leq$, see Example 2C2. Let $(X^*, \leq^*)$ be the antisymmetrization of $(X, \leq)$, i.e., the poset which is the quotient of $(X, \leq)$ under the following equivalence (1Bb):

$$x \sim x' \quad \text{iff both} \quad x \leq x' \quad \text{and} \quad x' \leq x \qquad (x, x' \in X).$$

Then the quotient map

$$\varphi\colon X \to X^* = X/\sim$$

has the following properties:

$$\begin{tikzcd}
(Y_i, \leq_i) \arrow[dr, "f_i"'] & {} \arrow[d] & \cdots \arrow[dl] & i \in I \\
& X \arrow[r, "\varphi"] \arrow[d, "h"] & (X/\sim, \leq^*) \arrow[dl, "h^*"] & \\
& (T, \leq) & &
\end{tikzcd}$$

(1) All $\varphi . f_i\colon (Y_i, \leq_i) \to (X^*, \leq^*)$ are order-preserving, i.e., morphisms in ***Pos*** $(i \in I)$;

(2) For each poset $(T, \leq)$ and each map $h\colon X \to T$ such that all $h . f_i\colon (Y_i, \leq_i) \to (T, \leq)$ are order-preserving $(i \in I)$, there exists a unique order-preserving map

$$h^*\colon (X^*, \leq^*) \to (T, \leq)$$

with

$$h = h^* . \varphi.$$

Proof. (1) is clear. For (2) we use the fact that since $\leq$ is the final preorder, the map

$$h\colon (X, \leq) \to (T, \leq)$$

is order-preserving. Then

$$x \sim x' \quad \text{implies} \quad h(x) = h(x') \qquad \text{for all} \quad x, x' \in X$$

(since $x \leq x'$ implies $h(x) \leq h(x')$, $x' \leq x$ implies $h(x') \leq h(x)$ and $\leq$ is antisymmetric). Thus, we can define a map $h^*\colon X/\sim \to T$ by

$$h^*([x]) = h(x) \qquad \text{for each} \quad x \in X.$$

This is the unique map with $h = h^* . \varphi$; and h^* is order-preserving since $[x] \leq^* [x']$ implies $x \leq x'$ and hence, $h^*([x]) \leq h^*([x'])$, for each $x, x' \in X$.

Remark. The poset $(X^*, \leq^*)$ has two properties analogous to those defining a final object. The basic difference is the fact that the underlying set is not the given set X but another set which is "connected" with X by a map $X \to X^*$; it is with respect to this map that these properties are formulated.

We are going to generalize this concept now.

2. Definition. A *semifinal object* of a sink

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$$

is an object (X^*, α^*) for which a map (called a *connecting map*)

$$\varepsilon: X \to X^*$$

with the following properties exists:

(1) all $\varepsilon . f_i: (Y_i, \beta_i) \to (X^*, \alpha^*)$ are morphisms, $i \in I$;

(2) for each object (T, δ) and each map $h: X \to T$ such that all $h . f_i: (Y_i, \beta_i) \to$ $\to (T, \delta)$ are morphisms, $i \in I$, there exists a unique morphism

$$h^*: (X^*, \alpha^*) \to (T, \delta)$$

with

$$h = h^* . \varepsilon .$$

$$\begin{tikzcd}
 & (Y_i, \beta_i) \arrow[d, "f_i"'] & \\
 & X \arrow[r, "\varepsilon"] \arrow[d, "h"'] & (X^*, \alpha^*) \arrow[dl, "h^*"] \\
 & (T, \delta) &
\end{tikzcd}$$

Remarks. (i) If $X^* = X$ and $\varepsilon = \mathrm{id}_X$ then (X^*, α^*) is the final object of the given sink: If all $h . f_i$ are morphisms then also h is a morphism, since $h = h^* . \varepsilon$ implies $h = h^*$. Thus, "semifinal" generalizes "final".

(ii) Let (X^*, α^*) be a semifinal object with a connecting map $\varepsilon: X \to X^*$. For arbitrary morphisms $h, k: (X^*, \alpha^*) \to (T, \delta)$

$$h . \varepsilon = k . \varepsilon \quad \text{implies} \quad h = k .$$

$$\begin{tikzcd}
X \arrow[r, "\varepsilon"] & (X^*, \alpha^*) \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & (T, \delta)
\end{tikzcd}$$

This follows from the uniqueness of h^* in the preceding definition.

Example: Vector spaces. Let $\{(Y_i, +, \cdot) \xrightarrow{f_i} X\}_{i \in I}$ be a sink of vector spaces. Let $(\hat{X}, +, \cdot)$ be a vector space with basis X. (For example, the vector space of all formal linear combinations $r_1 x_1 + \ldots + r_n x_n$ where $r_1, \ldots, r_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$ for some $n = 0, 1, 2, \ldots$.) Let $M_0 \subseteq \hat{X}$ denote the set of all vectors of the following type:

$$r\, f_i(y) + r'\, f_i(y') - f_i(ry + r'y')$$

for $i \in I$, $y, y' \in Y$ and $r, r' \in \mathbb{R}$. This set generates a subspace $M \subseteq X$, the linear span of M_0. Denote by

$$(X^*, +, \cdot)$$

the quotient space of $(X, +, \cdot)$ under the congruence $\sim_M$ (see 1Ga). Note that $\sim_M$ is the least congruence on $(X, +, \cdot)$ such that

$$(*) \qquad r\, f_i(y) + r'\, f_i(y') \sim f_i(ry + r'y')$$

holds for all $i \in I$, $y, y' \in Y$ and $r, r' \in \mathbb{R}$.

We claim that $(X^*, +, \cdot)$ is the semifinal vector space with the connecting map

$$\varepsilon: X \to X^*$$

defined as the restriction of the quotient map $\varphi: \hat{X} \to \hat{X}/\sim_M = X^*$, i.e.,

$$\varepsilon(x) = [x] \qquad \text{for each} \quad x \in X .$$

(1) All $\varepsilon . f_i: (Y_i, +, \cdot) \to (X^*, +, \cdot)$ are linear maps, $i \in I$. This follows immediately from $(*)$.

(2) Let $(T, +, \cdot)$ be a vector space and let $h: X \to T$ be a map such that all $h . f_i$ are linear, $i \in I$. We extend h to a linear map $\hat{h}: (\hat{X}, +, \cdot) \to (T, +, \cdot)$ by $\hat{h}(r_1 x_1 + \ldots + r_n x_n) = r_1\, h(x_1) + \ldots + r_n\, h(x_n)$ for each $r_1 x_1 + \ldots + r_n x_n \in \hat{X}$. For each vector

$$x = r\, f_i(y) + r\, f_i(y') - f_i(ry + ry')$$

in M_0 we use the linearity of $h . f_i$ to verify that $\hat{h}(x) = 0$:

$$\begin{aligned}
\hat{h}(x) &= r(\hat{h} . f_i)(y) + r'(\hat{h} . f_i)(y') - (\hat{h} . f_i)(ry + r'y') \\
&= r(h . f_i)(y) + r'(h . f_i)(y') - (h . f_i)(ry + r'y') \\
&= h . f_i(ry + r'y' - ry - r'y') \\
&= 0 .
\end{aligned}$$

Therefore, clearly,

$$x \in M \quad \text{implies} \quad \hat{h}(x) = 0 .$$

In other words,

$$x \sim_M x' \quad \text{implies} \quad \hat{h}(x) = \hat{h}(x') \qquad \text{for all} \quad x, x' \in \hat{X} .$$

Thus, we can define

$$h^*: X^* = \hat{X}/\sim_M \to T$$

by

$$h^*([x]) = \hat{h}(x) \qquad \text{for each} \quad x \in \hat{X} .$$

In particular,

$$h^*([x]) = h(x) \qquad \text{for each} \quad x \in X ,$$

hence,

$$h^* . \varepsilon = h .$$

The map h^* is linear because $h^* . \varphi = \hat{h}$ is linear. It is clear that h^* is the unique linear map with $h = h^* . \varepsilon$.

3. Definition. A transportable construct is said to be *semifinally complete* if each sink has a semifinal object.

Examples. (i) ***Vect*** is semifinally complete. It can be similarly proved that other algebraic constructs are semifinally complete. Given a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}$, we form the free algebra $(\hat{X}, \hat{\alpha})$ generated by X. Then we find the least congruence $\sim$ on X which "turns" all f_i into homomorphisms. The quotient object under this congruence is the semifinal algebra. In this way we can prove that the constructs

Mon, Sgr, Ab

(which we know to have free objects) are semifinally complete. We shall see later that also other algebraic constructs, e.g.,

Grp, Lat, Rng

are semifinally complete.

(ii) Each initially complete construct (***Gra, Top, Pros***) is also semifinally complete: by 2C2, each sink has a final object. We prove now, that all initially mono-complete constructs (***Pos***, $\boldsymbol{Met_k}$, $\boldsymbol{Top_2}$) are also semifinally complete.

4. Theorem. Every initially mono-complete construct is semifinally complete.

Proof. For each sink

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$$

consider the dual source (2C2)

$$\{X \xrightarrow{h_j} (T_j, \delta_j)\}_{j \in J} .$$

Denote by $\sim$ the following equivalence on X: given $x, x' \in X$,

$$x \sim x' \quad \text{iff} \quad h_j(x) = h_j(x') \qquad \text{for each} \quad j \in J .$$

As in the proof of 2A6 we factor

$$h_j = h'_j . \varphi \qquad (j \in J),$$

where $h'_j \colon X/\sim \to T$ sends each $[x]$ to $h_j(x)$. The source

$$\{X/\sim \xrightarrow{h'_j} (T_j, \delta_j)\}_{j \in J}$$

separates points and hence, it has an initial structure α. We claim that

$$(X/\sim, \alpha)$$

is the semifinal object of the given sink with respect to the quotient map

$$\varphi \colon X \to X/\sim .$$

$$\begin{tikzcd} (Y_i, \beta_i) \arrow[d, "f_i"'] & \\ X \arrow[r, "\varphi"] \arrow[dr, "h_j"'] & (X/\sim, \alpha) \arrow[d, "h'_j"] \\ & (T_j, \delta_j) \end{tikzcd}$$

(1) Each $\varphi . f_i \colon (Y_i, \beta_i) \to (X/\sim, \alpha)$ is a morphism, $i \in I$. To prove this it suffices to show that given $i \in I$, then all $h'_j . (\varphi . f_i) \colon (Y_i, \beta_i) \to (T_j, \delta_j)$ are morphisms, $j \in J$. This follows from

$$h'_j . \varphi . f_i = h_j . f_i$$

since the triple (T_j, δ_j, h_j) belongs to the dual source.

(2) Let (T, δ) be an object and $h \colon X \to T$ a map such that $h . f_i \colon (Y_i, \beta_i) \to (T, \delta)$ is a morphism for each $i \in I$. Then there exists $j \in J$ with $(T, \delta, h) = (T_j, \delta_j, h_j)$, and

$$h^* = h'_j \colon (X/\sim, \alpha) \to (T, \delta)$$

is a morphism with

$$h^* . \varphi = h .$$

This morphism is unique simply because φ is a surjection (thus, $h . \varphi = k . \varphi$ implies $h = k$). □

Remark. We have seen in the course of the preceding proof that initially mono-complete constructs have the following property: the semifinal object of each sink on X can be found on a quotient set $X/\sim$ (with the quotient map $\varphi \colon X \to X/\sim$ as the connecting map). This fact simplifies considerably the task of finding the semifinal object of a given sink.

The mentioned property actually characterizes initially mono-complete constructs, see Exercise c. below.

Example: the semifinal partial groupoids. Let $\{(Y_i, \circ_i) \xrightarrow{f_i} X\}_{i \in I}$ be a sink in $\boldsymbol{Grd_p}$. Since this construct is initially mono-complete, a semifinal partial groupoid on a quotient set of X can be found. Consider first an arbitrary equivalence $\sim$ on X and an arbitrary partial operation $*$ on $X/\sim$. If each $\varphi . f_i$ is a homomorphism, we see that

$$y' \circ_i y'' = y \quad \text{implies} \quad [f_i(y')] * [f_i(y'')] = [f_i(y)]$$

for all $i \in I$ and y, y', y'' in Y_i. Thus, given $j \in J$ and $z, z', z'' \in Y_j$ with $z' \circ_j z'' = z$, the following holds:

$$(*) \qquad f_i(y') \sim f_j(z') \ \text{ and } \ f_i(y'') \sim f_j(z'') \ \text{ imply } \ f_i(y) \sim f_j(z).$$

Thus, a candidate for the semifinal groupoid is determined as follows: let $\approx$ be the least equivalence on X with the property (*). It is easy to see that the meet of all equivalences satisfying (*) also satisfies it. Define an operation $\cdot$ on $X/\approx$ as follows:

$$[x'] \cdot [x''] = x \quad \text{iff} \quad x' = f_i(y'); \qquad x'' = f_i(y'') \quad \text{and} \quad x = f_i(y)$$

for some $i \in I$ and $y = y' \circ_i y''$ in Y.
It can be easily verified that

$$(X/\approx, \cdot)$$

is indeed a semifinal partial groupoid.

5. Definition. A construct $\mathscr{S}$ is said to be *trivial* if for each object (X, α) of $\mathscr{S}$ the set X has at most one point, i.e., if

$$\mathscr{S}[X] \neq \emptyset \quad \text{implies} \quad \operatorname{card} X \leqq 1 \qquad (X \text{ a set}).$$

All other constructs are called *non-trivial*.

Remark. A transformation monoid considered as a construct (1Da) is trivial iff its underlying set has at most one point (and only the identity transformation is considered). With this exception, all the constructs mentioned in the preceding sections are non-trivial.

The reason for introducing the concept of triviality is to obtain free objects as special semifinal objects. Recall that the final object of the empty sink is the discrete object (2C3); now we characterize the semifinal objects.

Proposition. In a non-trivial construct, free objects are precisely the semifinal objects of the empty sinks.

Proof. I. Let X be a set and let (X^*, α^*) be the semifinal object of the empty sink on X. Let $\varepsilon: X \to X^*$ denote the connecting map. We shall prove that (X^*, α^*) is free over the subset $M = \varepsilon(X)$. The semifinality means that for each object (T, δ)

$$\begin{tikzcd}
X \arrow[rr, "\varepsilon"] \arrow[dr, "\varepsilon_0"] \arrow[dd, "h"'] & & (X^*, \alpha^*) \arrow[ddll, "h^*"] \\
& M \arrow[ur, "v"] \arrow[dl, "k"] & \\
(T, \delta) & &
\end{tikzcd}$$

and each map $h: X \to T$ there exists a unique morphism $h^*: (X^*, \alpha^*) \to (T, \delta)$ such that $h = h^* . \varepsilon$. Thus, if $X \subseteqq X^*$ and ε is the inclusion map then (X^*, α^*) is free over X (since $h = h^* . \varepsilon$ then means that h^* extends h). For a general ε first observe that ε is one-to-one: choose an object (T, δ) with two distinct points $t, t' \in T$. (This is possible since our constructs is non-trivial.) For arbitrary

$$x, x' \in X \quad \text{with} \quad x \neq x'$$

choose any map $h: X \to T$ such that $h(x) = t$ and $h(x') = t'$. The morphism h^* fulfils $h = h^* . \varepsilon$ – thus, $h(x) \neq h(x')$ implies

$$\varepsilon(x) \neq \varepsilon(x').$$

Now, we shall prove that (X^*, α^*) is free over

$$M = \varepsilon(X).$$

First, denote by

$$\varepsilon_0: X \to M$$

the bijection which is the range-restriction of ε; i.e.,

$$\varepsilon = v . \varepsilon_0$$

where $v: M \to X$ is the inclusion map. For each object (T, δ) and each map $k: M \to T$ put

$$h = k . \varepsilon_0 : X \to T.$$

There exists a unique morphism $h^*: (X^*, \alpha^*) \to (T, \delta)$ with

$$h = h^* . \varepsilon = (h^* . v) . \varepsilon_0.$$

Then h^* extends k since

$$k = (k . \varepsilon_0) . \varepsilon_0^{-1} = h . \varepsilon_0^{-1} = (h^* . v) . \varepsilon_0 . \varepsilon_0^{-1} = h^* . v.$$

II. Let (X^*, α^*) be a free object over $X \subseteqq X^*$. Then (X^*, α^*) is the semifinal object of the empty sink on X with the inclusion map $v: X \to X^*$ being the connecting map. Indeed, for each object (T, δ) and each map $h: X \to T$ there exists a unique extension to a morphism $h^*: (X^*, \alpha^*) \to (T, \delta)$, i.e., a unique morphism with $h = h^* . v$. □

Corollary. Each non-trivial, semifinally complete construct has free objects.

6. Since free objects are not unique, but only unique up to an isomorphism (1H3), it follows that semifinal objects are also not unique. We shall prove that they too are unique up to an isomorphism. The proof is analogous to that for the free objects.

Proposition. Let (X^*, α^*) be a semifinal object of a sink. Then
(i) each other semifinal object of this sink is isomorphic to (X^*, α^*);
(ii) each object isomorphic to (X^*, α^*) is semifinal with respect to the given sink.

Proof. Let $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i\in I}$ be a sink, and let (X^*, α^*) be a semifinal object with a connecting map $\varepsilon\colon X \to X^*$.

(i) If (X^+, α^+) is another semifinal object with a connecting map $\tilde{\varepsilon}$, we prove that $(X^*, \alpha^*) \cong (X^+, \alpha^+)$.

$$\begin{tikzcd}
(Y_i,\beta_i) \arrow[d, "f_i"] & \\
X \arrow[r, "\varepsilon"] \arrow[d, "\tilde{\varepsilon}"'] & (X^*,\alpha^*) \arrow[dl, shift left, "\tilde{\varepsilon}^*"] \\
(X^+,\alpha^+) \arrow[ur, shift left, "\varepsilon^+"] &
\end{tikzcd}$$

For each $i \in I$,

$$\tilde{\varepsilon} . f_i\colon (Y_i, \beta_i) \to (X^+, \alpha^+)$$

is a morphism and hence, there exists a unique morphism

$$\tilde{\varepsilon}^*\colon (X^*, \alpha^*) \to (X^+, \alpha^+)$$

with

$$\tilde{\varepsilon} = \tilde{\varepsilon}^* . \varepsilon .$$

This follows from the semifinality of (X^*, α^*). Analogously, there exists a unique morphism

$$\varepsilon^+\colon (X^+, \alpha^+) \to (X^*, \alpha^*)$$

with

$$\varepsilon = \varepsilon^+ . \tilde{\varepsilon} .$$

It suffices to show that $\tilde{\varepsilon}^*$ and ε^+ are inverse to each other. Note that for the map $h = \varepsilon$ we have a unique (!) morphism $h^*\colon (X^*, \alpha^*) \to (X^*, \alpha^*)$ such that $\varepsilon = h^* . \varepsilon$; by the uniqueness, $h^* = \mathrm{id}_{X^*}$. Now, the morphism

$$\varepsilon^+ . \tilde{\varepsilon}^*\colon (X^*, \alpha^*) \to (X^*, \alpha^*)$$

fulfils

$$(\varepsilon^+ . \tilde{\varepsilon}^*) . \varepsilon = \varepsilon^+ . \tilde{\varepsilon} = \varepsilon$$

and hence,

$$\varepsilon^+ . \tilde{\varepsilon}^* = \mathrm{id}_{X^*} .$$

Similarly,

$$\tilde{\varepsilon}^* . \varepsilon^+ = \mathrm{id}_{X^+} .$$

(ii) Let $p\colon (X^*, \alpha^*) \to (X^+, \alpha^+)$ be an isomorphism.

$$\begin{tikzcd}
(Y_i,\beta_i) \arrow[d, "f_i"] & & \\
X \arrow[r, "\varepsilon"] \arrow[d, "h"'] & (X^*,\alpha^*) \arrow[r, shift left, "p"] \arrow[dl, "h^*"] & (X^+,\alpha^+) \arrow[l, shift left, "p^{-1}"] \\
(T,\delta) & &
\end{tikzcd}$$

We shall prove that (X^+, α^+) is semifinal, with the connecting map $\tilde{\varepsilon} = p . \varepsilon$. For each object (T, δ) and each map $h\colon X \to T$ such that all $h . f_i$ are morphisms, there exists a unique morphism

$$h^*\colon (X^*, \alpha^*) \to (T, \delta)$$

with $h = h^* . \varepsilon$. Then the morphism

$$h^+ = h^* . p^{-1}\colon (X^+, \alpha^+) \to (T, \delta)$$

fulfils

$$h = h^* . p^{-1} . p . \varepsilon = h^+ . \tilde{\varepsilon} .$$

And h^+ is unique: if $k\colon (X^+, \alpha^+) \to (T, \delta)$ also fulfils $h = k . \tilde{\varepsilon}$ then $h = (k . p) . \varepsilon$; therefore,

$$k . p = h^*$$

i.e.,

$$k = h^* . p^{-1} = h^+ .$$

Thus, (X^+, α^+) is semifinal. □

7. Concluding remark. Semifinal completeness is a property of constructs which we often meet in algebra, topology and elsewhere. We present a criterion for semifinal completenes in the next section. This is also a useful criterion for the existence of free objects.

The other way round, the constructs which fail to have free objects (***Met***, ***Clat***, etc.) are not semifinally complete. Semifinal completeness can be viewed as a property characteristic of the "well-behaved" constructs.

Exercises 2D

a. Semifinal algebras. (1) Describe the semifinal monoids and prove that ***Mon*** is semifinally complete. Hint: this is analogous to Example 2D2; here $\hat{X} = X^*$ will be the word-monoid over X and $\sim$ will be the least congruence on X^* such that the two-letter words $f_i(y)\, f_i(y')$ are congruent to the one-letter words $f_i(y \circ_i y')$ (for all $i \in I$ and $y, y' \in Y_i$). Then the quotient monoid is semifinal.

(2) Check that the situation in ***Sgr*** is analogous.

(3) Describe the semifinal groups in ***Ab***. Hint: similar to (1); denote by $\hat{X}$ the free Abelian group, see 1Hb.

b. Semifinal topologies and metrics. (1) Describe semifinal objects in $\boldsymbol{Top_0}$, $\boldsymbol{Top_1}$, $\boldsymbol{Top_2}$. Hint: use the final topologies and Remark 2D4.

(2) Prove that in $\boldsymbol{Met_k}$ the semifinal metric of a sink $\{(Y_i, \beta_i) \xrightarrow{f_i} X\}$ is obtained as follows: if α is the final pseudometric (2Cb) then $X^* = X/\sim$ where $x \sim x'$ iff $\alpha(x, x') = 0$ and $\alpha^*([x], [y]) = \alpha(x, y)$ for all $x, y \in X$.

c. Initially mono-complete constructs. Prove the following converse to Remark 2D4: if each sink in a transportable construct has a semifinal object with a surjective connecting map, then the construct is initially mono-complete. Hint: use dual sinks.

2E. A Criterion for Semifinal Completeness

1. Theorem. Each semifinally complete construct has Cartesian products and intersections.

Proof. Let $\mathscr{S}$ be a semifinally complete construct. Let (T_j, δ_j), $j \in J$, be its objects. To show that $\mathscr{S}$ has Cartesian products, put

$$X = \prod_{j \in J} T_j$$

and denote by $p_j\colon X \to T_j$ the projections. It is our task to show that the following source

$$(*) \qquad \{X \xrightarrow{p_j} (T_j, \delta_j)\}_{j \in J}$$

has an initial structure. To show that $\mathscr{S}$ has intersections, assume that each (T_j, δ_j) is a subobject of a given object $(\tilde{T}, \tilde{\delta})$; put

$$X = \bigcap_{j \in J} T_j$$

and denote by $p_j\colon X \to T_j$ the inclusion maps. Again, it suffices to show that the source (*) has an initial structure α: then (X, α) is the intersection of the objects (T_j, δ_j). Indeed, if $v\colon X \to \tilde{T}$ and $v_j\colon T_j \to \tilde{T}$ denote the inclusion maps, then

$$v = v_j \cdot p_j \qquad \text{for each} \quad j \in J\,.$$

Hence, $v\colon (X, \alpha) \to (\tilde{T}, \tilde{\delta})$ is a morphism. Moreover, given an object (Z, γ) and a map $h\colon Z \to X$ such that $v \cdot h = v_j \cdot (p_j \cdot h)\colon (Z, \gamma) \to (\tilde{T}, \tilde{\delta})$ is a morphism, then each $p_j \cdot h$ is a morphism and hence, $h\colon (Z, \gamma) \to (X, \alpha)$ is a morphism.

To prove that (*) has an initial structure let

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$$

denote its dual source (2C2), and let (X^*, α^*) be the semifinal object with connecting map

$$\varepsilon\colon X \to X^*\,.$$

We shall prove that ε is a bijection. Then it transports the structure α^* to a structure α such that

$$\varepsilon\colon (X, \alpha) \to (X^*, \alpha^*)$$

is an isomorphism (since $\mathscr{S}$ is transportable by the definition of semifinal completeness). It is then easy to see that α is the initial structure of (*). Thus, the proof will be concluded when we exhibit a map $\tilde{\varepsilon}\colon X^* \to X$ inverse to ε.

$$\begin{tikzcd}
(Y_i, \beta_i) \arrow[d, "f_i"'] & & \\
X \arrow[rr, shift left, "\varepsilon"] \arrow[dd, "p_j"'] & & (X^*, \alpha^*) \arrow[ll, shift left, "\tilde{\varepsilon}"] \arrow[ddll, "p_j^*"] \\
& & \\
(T_j, \delta_j) \arrow[rr, dashed] & & (\tilde{T}, \tilde{\delta})
\end{tikzcd}$$

Given $j \in J$, all $p_j \cdot f_i$ are morphisms and thus we have a unique morphism $p_j^*\colon (X^*, \alpha^*) \to (T_j, \delta_j)$ with

$$(1) \qquad p_j = p_j^* \cdot \varepsilon \qquad (j \in J)\,.$$

There is a unique map $\tilde{\varepsilon}\colon X^* \to X$ with

$$(2) \qquad p_j^* = p_j \cdot \tilde{\varepsilon} \qquad (j \in J)\,.$$

This is clear in the case of products: define $\tilde{\varepsilon}$ by $\tilde{\varepsilon}(x) = \{p_j^*(x)\}_{j \in J}$ for all $x \in X$. In the case of intersections, the map $v\colon X \to \tilde{T}$ has the property that each $v \cdot f_i = v_j \cdot p_j \cdot f_i\colon (Y_i, \beta_i) \to (\tilde{T}, \tilde{\delta})$ is a morphism, hence, there is a unique (!) morphism $v^*\colon (X^*, \alpha^*) \to (\tilde{T}, \tilde{\delta})$ with $v = v^* \cdot \varepsilon$. For each $j \in J$, the morphism $v_j \cdot p_j^*$ fulfils $(v_j \cdot p_j^*)\,\varepsilon = v_j \cdot p_j = v$ and therefore,

$$v_j \cdot p_j^* = v^* \qquad \text{for all} \quad j \in J\,.$$

Hence, for each $x \in X^*$ the point $p_j^*(x)$ is independent of j and it lies in T_j – thus, it lies in $X = \bigcap T_j$. We can define $\tilde{\varepsilon}\colon X^* \to X$ by

$$\tilde{\varepsilon}(x) = p_j^*(x) \qquad \text{for each} \quad x \in X, \quad j \in J\,.$$

Then (2) holds.

To prove that $\bar{\varepsilon}$ is inverse to ε, first use the equality

$$p_j \cdot (\bar{\varepsilon} \cdot \varepsilon) = p_j^* \cdot \varepsilon = p_j \qquad (\text{for all } j \in J)$$

to conclude that

$$\bar{\varepsilon} \cdot \varepsilon = \mathrm{id}_X$$

(which follows immediately both for products and for intersections). Next, the map $(X^*, \alpha^*) \xrightarrow{\bar{\varepsilon}} X$ belongs to the dual sink of (*) because each $p_j \cdot \bar{\varepsilon} = p_j^*, j \in J$, is a morphism. Hence, $\varepsilon \cdot \bar{\varepsilon}$: $(X^*, \alpha^*) \to (X^*, \alpha^*)$ is also a morphism. Since

$$(\varepsilon \cdot \bar{\varepsilon}) \cdot \varepsilon = \varepsilon \cdot (\bar{\varepsilon} \cdot \varepsilon) = \varepsilon ,$$

we conclude that

$$\varepsilon \cdot \bar{\varepsilon} = \mathrm{id}_{X^*},$$

see Remark 2D2(ii). □

2. If we are to decide whether a certain construct is semifinally complete, we should first check for Cartesian products (which is usually easy). Then we should study the generation of subobjects (1F5).

Definition. A construct with intersections is said to *have bounded generation* if for each cardinal n there is a cardinal n^* such that each object on n generators has at most n^* points (i.e., if (X, α) has n generators then $\operatorname{card} X \leq n^*$).

The following elementary properties of cardinals are used in the subsequent examples.

(i) For each infinite set X

$$\operatorname{card} X = \operatorname{card} X \times X .$$

(i) If $\operatorname{card} X = n$ then $\operatorname{card}(\exp X)$ is larger than n; it is denoted by 2^n (since it is the cardinality of the set $\{0, 1\}^X$). We have

$$\operatorname{card} \mathbb{R} = 2^{\aleph_0} .$$

(iii) For each infinite set X,

$$\operatorname{card} X = \operatorname{card} \{M \subseteq X ; M \text{ finite}\} .$$

(iv) If X and Y are disjoint sets then $\operatorname{card}(X \cup Y)$ is denoted by $n + m$, where $\operatorname{card} X = n$ and $\operatorname{card} Y = m$. If n is infinite, then

$$n + \aleph_0 = n ;$$

if n is finite, then

$$n + \aleph_0 = \aleph_0 .$$

(v) If infinite sets $X_0, X_1, X_2, \ldots$ have the same cardinality n, then also $\operatorname{card}(\bigcup_{k=0}^{\infty} X_k) = n$.

Examples. (i) ***Sgr*** has bounded generation with

$$n^* = n + \aleph_0 \qquad \text{for each cardinal } n .$$

In fact, let $(X, \cdot)$ be a semigroup, generated by a set $M \subseteq X$ with $\operatorname{card} M \leq n$. Put $M_0 = M$ and

$$M_1 = \{x \cdot y;\ x, y \in M_0\} \cup M_0 ,$$
$$M_2 = \{x \cdot y;\ x, y \in M_1\} \cup M_1 ,$$

etc. The set $\bar{M} = \bigcup_{i=0}^{\infty} M_i$ is a subsemigroup: given $x, y \in \bar{M}$ there is an i with $x, y \in M_i$; then $x \cdot y \in M_{i+1}$. Since $M \subseteq \bar{M}$, we conclude that $X = \bar{M}$. Moreover, if M is finite then each M_i is finite, hence, $\operatorname{card} \bar{M} \leq \aleph_0$; if M is infinite then $M_0, M_1 \ldots$ have the same cardinality as M. Therefore, $\operatorname{card} X = \operatorname{card} \bar{M} \leq n + \aleph_0$.

(ii) Other algebraic constructs, e.g.,

Mon, Grp, Rng, Lat

have bounded generation with $n^* = n + \aleph_0$ for all n – the proof is similar to (i).

(iii) The construct ***Vect*** has bounded generation with $n^* = n + 2^{\aleph_0}$. If a vector space has dimension $\leq n$, it has a basis M of cardinality $\leq n$. All elements are linear combinations $\sum_{i=1}^{k} r_i x_i$ $(x_i \in M)$. The number of all linear combinations is clearly

$$\operatorname{card} \mathbb{R} \times \operatorname{card} M \leq n + 2^{\aleph_0} .$$

(iv) All hereditary constructs have bounded generation with

$$n^* = n \qquad \text{for each cardinal } n .$$

In fact, an object (X, α) is generated by $M \subseteq X$ only if $M = X$, then $\operatorname{card} M = \operatorname{card} X$. Thus, ***Top, Pos, Met***, etc., have (trivially) bounded generation.

(v) The construct ***Comp*** (of compact T_2-spaces) has bounded generation with

$$n^* = 2^{2^n} \qquad \text{for each cardinal } n .$$

Indeed, let (X, α) be a space generated in ***Comp*** by $M \subseteq X$. Then M is a dense subset of X (1Ff). For each point $x \in X$ put

$$A_x = \{U \subseteq M;\ x \in \bar{U}\} .$$

Then $A_x \subseteq \exp M$, and

$$x \neq x' \quad \text{implies} \quad A_x \neq A_{x'} \qquad \text{for all} \quad x, x' \in X .$$

(If $x \neq x'$ then there exist disjoint open sets U, V with $x \in U$ and $x' \in V$. Since M is dense, we have $U \cap M \in A_x$, but $U \cap M \notin A_{x'}$ since $V \cap (U \cap M) = \emptyset$.)

Thus, the number of points in X is smaller than or equal to the number of subsets of $\exp M$. Hence,

$$\operatorname{card} M \leqq n \quad \text{implies} \quad \operatorname{card} X \leqq 2^{2^n}.$$

Remark. An example of a construct without bounded generation is ***Clat***: there exist arbitrarily large complete lattices even on three generators. Another construct without bounded generation is ***Topc***: this construct does not have intersections, hence, it does not have bounded generation by definition (recall that it does not have generation (1F4 and 1Fb)).

3. Theorem. Each fibre-small construct with Cartesian products and bounded generation is semifinally complete.

Proof. For each sink

$$\{(Y_i, \beta_i) \xrightarrow{f_i} X\}_{i \in I}$$

we are going to find a semifinal object. Put

$$n = \operatorname{card} X;$$

there is a cardinal n^* such that each object on n generators has at most n^* points. For every cardinal $k \leqq n$ choose a set T_k with

$$\operatorname{card} T_k = k.$$

Let us consider all objects (T_k, δ), $k \leqq n$ and $\delta \in \mathscr{S}[T_k]$, with the following property:

(*) there exists a map $\varepsilon: X \to T_k$ such that all $\varepsilon . f_i: (Y_i, \beta_i) \to (T_k, \delta)$ are morphisms $(i \in I)$.

$$\begin{tikzcd}
(Y_i, \beta_i) \arrow[dr, "f_i"'] & {} \arrow[d] & \cdots \arrow[dl] & \\
& X \arrow[r, "\varepsilon"] & (T_k, \delta) &
\end{tikzcd}$$

All these objects form a set — a subset of

$$\bigcup_{k \leqq n^*} \mathscr{S}[T_k]$$

(which is a set since the construct $\mathscr{S}$ is fibre-small!). Hence, all the triples $(T_k, \delta, \varepsilon)$, where (T_k, δ) is an object satisfying (*) and $\varepsilon: X \to T_k$ is the corresponding map, can be written as a collection

$$(T_{k(j)}, \delta_j, \varepsilon_j), \qquad j \in J,$$

where J is a set.

Let us form the Cartesian product of the objects $(T_{k(j)}, \delta_j)$:

$$(\tilde{X}, \tilde{\alpha}) = \prod_{j \in J} (T_{k(j)}, \delta_j).$$

Define a map

$$\tilde{\varepsilon}: X \to \tilde{X} = \prod_{j \in J} T_{k(j)}$$

as follows:

$$\tilde{\varepsilon}(x) = \{\varepsilon_j(x)\}_{j \in J} \qquad (x \in X),$$

i.e., by

$$\pi_j . \tilde{\varepsilon} = \varepsilon_j \qquad \text{for each} \quad j \in J.$$

The set

$$M = \tilde{\varepsilon}(X) \subseteqq \tilde{X}$$

generates a subobject

$$(X^*, \alpha^*)$$

of $(\tilde{X}, \tilde{\alpha})$. Denote by

$$\varepsilon^*: X \to X^*$$

the restriction of $\tilde{\varepsilon}$; the inclusion map $v: X^* \to \tilde{X}$ fulfils

$$v . \varepsilon^* = \tilde{\varepsilon}.$$

We are going to prove that (X^*, α^*) is a semifinal object of the given sink with the connecting map ε^*.

$$\begin{tikzcd}
(Y_i, \beta_i) \arrow[d, "f_i"] & & \\
X \arrow[r, "\varepsilon^*"] \arrow[d, "\varepsilon_j"] & (X^*, \alpha^*) \arrow[r, "v"] & (\tilde{X}, \tilde{\alpha}) \arrow[dll, "\pi_j"] \\
(T_{k(j)}, \delta_j) & &
\end{tikzcd}$$

(1) For each $i \in I$

$$\varepsilon^* . f_i: (Y_i, \beta_i) \to (X^*, \alpha^*)$$

is a morphism. By the property (*), all

$$\varepsilon_j . f_i = \pi_j . (\tilde{\varepsilon} . f_i): (Y_i, \beta_i) \to (T_{k(j)}, \delta_j) \qquad (j \in J)$$

are morphisms. By the definition of Cartesian product, this implies that

$$\tilde{\varepsilon} . f_i \colon (Y_i, \beta_i) \to (\tilde{X}, \tilde{\alpha})$$

is a morphism. Since $\tilde{\varepsilon} = v . \varepsilon^*$, by the definition of subobject also

$$\varepsilon^* . f_i \colon (Y_i, \beta_i) \to (X^*, \alpha^*)$$

is a morphism.

(2) Let (T, δ) be an object and $h \colon X \to T$ a map such that all $h . f_i \colon (Y_i, \beta_i) \to$ $\to (T, \delta)$, $i \in I$, are morphisms. The set $h(X) \subseteq T$ generates a subobject (T', δ') of (T, δ); denote by

$$h' \colon X \to T'$$

the restriction of h, i.e., the map such that

$$h = w . h'$$

for the inclusion map $w \colon T' \to T$.

Since

$$\operatorname{card} h(X) \leqq \operatorname{card} X = n,$$

the object (T', δ') has n generators which implies

$$\operatorname{card} T' \leqq n^* .$$

Therefore, there exists a cardinal $k \leqq n^*$ such that T' and T_k are isomorphic sets. Choose any bijection

$$b \colon T' \to T_k .$$

$$\begin{tikzcd}
(Y_i, \beta_i) \arrow[d, "f_i"] & & \\
X \arrow[r, "\varepsilon^*"] \arrow[dr, "h'"] \arrow[drr, "\hat{\varepsilon} = \varepsilon_{j_0}"] \arrow[ddr, "h"'] & (X^*, \alpha^*) \arrow[r, "v"] & (\tilde{X}, \tilde{\alpha}) \arrow[d, "\pi_{j_0}"] \\
& (T', \delta') \arrow[r, "b"] \arrow[d, "w"] & (T_k, \hat{\delta}) = (T_{k(j_0)}, \delta_{j_0}) \\
& (T, \delta) &
\end{tikzcd}$$

Denote by $\hat{\delta}$ the structure, transported by b, i.e., such that

$$b \colon (T', \delta') \to (T_k, \hat{\delta})$$

is an isomorphism. Then the object $(T_k, \hat{\delta})$ has property (*) with respect to

$$\hat{\varepsilon} = b . h' = X \to T_k .$$

Indeed, $h . f_i = w . (h' . f_i) \colon (Y_i, \beta_i) \to (T, \delta)$ is a morphism for each $i \in I$. Therefore, $h' . f_i \colon (Y_i, \beta_i) \to (T', \delta')$ is a morphism, and so is

$$\hat{\varepsilon} . f_i = b . (h' . f_i) \colon (Y_i, \beta_i) \to (T_k, \hat{\delta}) .$$

This implies that there exist $j_0 \in J$ with

$$(T_k, \hat{\delta}, \hat{\varepsilon}) = (T_{k(j_0)}, \delta_{j_0}, \varepsilon_{j_0}) .$$

Put

$$h^* = w . b^{-1} . \pi_{j_0} . v \colon (X^*, \alpha^*) \to (T, \delta) .$$

This is a morphism since each of the maps composing h^* is a morphism. And

$$\begin{aligned}
h &= w . h' \\
&= w . b^{-1} . \hat{\varepsilon} && [\hat{\varepsilon} = b . h'] \\
&= w . b^{-1} . \pi_{j_0} . \tilde{\varepsilon} && [\hat{\varepsilon} = \varepsilon_{j_0} = \pi_{j_0} . \tilde{\varepsilon}] \\
&= w . b^{-1} . \pi_{j_0} . v . \varepsilon^* && [\tilde{\varepsilon} = v . \varepsilon^*] \\
&= h^* . \varepsilon^* .
\end{aligned}$$

To prove that h^* is unique, let $k \colon (X^*, \alpha^*) \to (T, \delta)$ be another morphism with

$$k . \varepsilon^* = h .$$

By Proposition 2B7, the set

$$E = \{x \in X^* ;\ k(x) = h^*(x)\}$$

is a subobject of (X^*, α^*), hence, of $(\tilde{X}, \tilde{\alpha})$ (see 1Fa). Since $h^* . \varepsilon^* = k . \varepsilon^*$, we have

$$E \supseteq \varepsilon^*(X) = \tilde{\varepsilon}(X) = M .$$

Since M generates (X^*, α^*), this implies

$$E = X^* .$$

Therefore, $k(x) = h^*(x)$ for each $x \in X^*$, in other words, $k = h^*$. □

4. Corollary. Each non-trivial, fibre-small construct with Cartesian products and bounded generation has free objects.

This follows from Corollary 2D5.

Examples. The following constructs have free objects:

***Lat*, *Grd*, *Grp*, *Rng*, *Comp*.**