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# Preface

This book is intended for people who are not primarily algebraists, but nonetheless get involved in the subject through other areas – having backgrounds, in say, engineering, physics, geometry or analysis. Readers of this book may have different starting levels, backgrounds and goals.

**Chapters 1–2** form a unit for an undergraduate course: Vectors, the scalar product; complex numbers, a geometrical interpretation of the imaginary unit  $i = \sqrt{-1}$ .

**Chapters 3–7** guide the reader through bottlenecks and provide necessary building blocks: Bivectors and the exterior product. Pauli spin matrices and Pauli spinors. Quaternions and the fourth dimension. The cross product is generalized to higher dimensions.

**Chapters 8–10** aim to serve readers with different backgrounds: Electromagnetism, special relativity, Dirac theory. The Dirac equation is formulated with complex column spinors, spinors in minimal left ideals and considering spinors as operators.

**Chapters 11–13** discuss physical applications of spinors: In the case of an electron, Fierz identities are sufficient to reconstruct spinors from their physical observables, but this is not the case for the neutrino. Boomerangs are introduced to handle neutrinos. A new class of spinors is identified by its bilinear observables: the *flag-dipole spinors* which reside between Weyl, Majorana and Dirac spinors.

**Chapters 14–15** are more algebraic than the previous chapters. Clifford algebras are defined for the first time. Finite fields. Isometry classes of quadratic forms and their Witt rings. Tensor products of algebras and Brauer groups are discussed.

**Chapters 16–19** view Clifford algebras through matrix algebras: Clifford algebras are given isomorphic images as matrix algebras, Cartan's periodicity of 8, spin groups and their matrix images in lower dimensions, scalar products of spinors with a chessboard of their automorphism groups, Möbius transformations represented by Vahlen matrices.

**Chapters 20–23** discuss miscellaneous mathematical topics. A one-variable higher-dimensional generalization of complex analysis: Cauchy's integral formula is generalized to higher dimensions. Multiplication rule of standard basis elements of a Clifford algebra and its relation to Walsh functions. Multivector structure of Clifford algebras and the