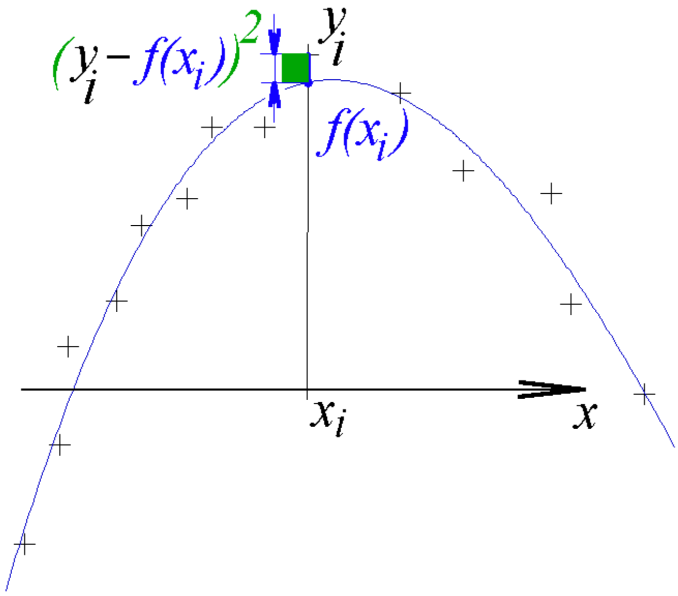
Aproximace funkce metodou nejmenších čtverců



Vstupní data zatížena chybou
(např. naměřené hodnoty)

$$\cancel{\sum_i \left(y_i - f(x_i)\right) = \min}$$

$$\cancel{\sum_i \left|y_i - f(x_i)\right| = \min}$$

$$\sum_i \left(y_i - f(x_i)\right)^2 = \min$$

$i = 0;1;\ldots;n$

$y_i - f(x_i)$ reziduum

$$f(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x)$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$\varphi_0(x) = 1; \varphi_1(x) = x; \varphi_2(x) = x^2;$$

$\cancel{m = n \Rightarrow f(x)}$ ~~prochází všemi body~~

$m < n$

$$f(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + a_2\varphi_2(x)$$

$$\sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right]^2 = min$$

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$$\sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right]^2 = min$$

$$H(a_0; a_1; a_2) = \sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right]^2 = min$$

$$\frac{\partial H(a_0; a_1; a_2)}{\partial a_0} = 2\sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right] \cdot \left(-\varphi_0(x_i)\right) = 0$$

$$\frac{\partial H(a_0; a_1; a_2)}{\partial a_1} = 2\sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right] \cdot \left(-\varphi_1(x_i)\right) = 0$$

$$\frac{\partial H(a_0; a_1; a_2)}{\partial a_2} = 2\sum_i \left[y_i - \left(a_0\varphi_0(x_i) + a_1\varphi_1(x_i) + a_2\varphi_2(x_i)\right)\right] \cdot \left(-\varphi_2(x_i)\right) = 0$$

$$\sum_i \left[-y_i\varphi_0(x_i) + a_0\varphi_0(x_i)\varphi_0(x_i) + a_1\varphi_0(x_i)\varphi_1(x_i) + a_2\varphi_0(x_i)\varphi_2(x_i)\right] = 0$$

$$\sum_i \left[-y_i\varphi_1(x_i) + a_0\varphi_1(x_i)\varphi_0(x_i) + a_1\varphi_1(x_i)\varphi_1(x_i) + a_2\varphi_1(x_i)\varphi_2(x_i)\right] = 0$$

$$\sum_i \left[-y_i\varphi_2(x_i) + a_0\varphi_2(x_i)\varphi_0(x_i) + a_1\varphi_2(x_i)\varphi_1(x_i) + a_2\varphi_2(x_i)\varphi_2(x_i)\right] = 0$$

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$$\sum_i [-y_i\varphi_0(x_i) + a_0\varphi_0(x_i)\varphi_0(x_i) + a_1\varphi_0(x_i)\varphi_1(x_i) + a_2\varphi_0(x_i)\varphi_2(x_i)] = 0$$
$$\sum_i [-y_i\varphi_1(x_i) + a_0\varphi_1(x_i)\varphi_0(x_i) + a_1\varphi_1(x_i)\varphi_1(x_i) + a_2\varphi_1(x_i)\varphi_2(x_i)] = 0$$
$$\sum_i [-y_i\varphi_2(x_i) + a_0\varphi_2(x_i)\varphi_0(x_i) + a_1\varphi_2(x_i)\varphi_1(x_i) + a_2\varphi_2(x_i)\varphi_2(x_i)] = 0$$

$$\sum_i (-y_i\varphi_0(x_i)) + \sum_i (a_0\varphi_0(x_i)\varphi_0(x_i)) + \sum_i (a_1\varphi_0(x_i)\varphi_1(x_i)) + \sum_i (a_2\varphi_0(x_i)\varphi_2(x_i)) = 0$$
$$\sum_i (-y_i\varphi_1(x_i)) + \sum_i (a_0\varphi_1(x_i)\varphi_0(x_i)) + \sum_i (a_1\varphi_1(x_i)\varphi_1(x_i)) + \sum_i (a_2\varphi_1(x_i)\varphi_2(x_i)) = 0$$
$$\sum_i (-y_i\varphi_2(x_i)) + \sum_i (a_0\varphi_2(x_i)\varphi_0(x_i)) + \sum_i (a_1\varphi_2(x_i)\varphi_1(x_i)) + \sum_i (a_2\varphi_2(x_i)\varphi_2(x_i)) = 0$$

$$-\sum_i (y_i\varphi_0(x_i)) + a_0\sum_i (\varphi_0(x_i)\varphi_0(x_i)) + a_1\sum_i (\varphi_0(x_i)\varphi_1(x_i)) + a_2\sum_i (\varphi_0(x_i)\varphi_2(x_i)) = 0$$
$$-\sum_i (y_i\varphi_1(x_i)) + a_0\sum_i (\varphi_1(x_i)\varphi_0(x_i)) + a_1\sum_i (\varphi_1(x_i)\varphi_1(x_i)) + a_2\sum_i (\varphi_1(x_i)\varphi_2(x_i)) = 0$$
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$$\mathbf{u} = (u_1; u_2; \dots; u_m); \mathbf{v} = (v_1; v_2; \dots; v_m); \qquad \mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_mv_m = \sum_{i=1}^{m}(u_iv_i)$$

Aproximace funkce metodou nejmenších čtverců

$$-\mathbf{y}\cdot\boldsymbol{\varphi_0} + a_0\cdot\boldsymbol{\varphi_0}\cdot\boldsymbol{\varphi_0} + a_1\cdot\boldsymbol{\varphi_0}\cdot\boldsymbol{\varphi_1} + a_2\cdot\boldsymbol{\varphi_0}\cdot\boldsymbol{\varphi_2} = 0$$
$$-\mathbf{y}\cdot\boldsymbol{\varphi_1} + a_0\cdot\boldsymbol{\varphi_1}\cdot\boldsymbol{\varphi_0} + a_1\cdot\boldsymbol{\varphi_1}\cdot\boldsymbol{\varphi_1} + a_2\cdot\boldsymbol{\varphi_1}\cdot\boldsymbol{\varphi_2} = 0$$
$$-\mathbf{y}\cdot\boldsymbol{\varphi_2} + a_0\cdot\boldsymbol{\varphi_2}\cdot\boldsymbol{\varphi_0} + a_1\cdot\boldsymbol{\varphi_2}\cdot\boldsymbol{\varphi_1} + a_2\cdot\boldsymbol{\varphi_2}\cdot\boldsymbol{\varphi_2} = 0$$

$$-(\mathbf{y};\boldsymbol{\varphi_0}) + a_0\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) = 0$$
$$-(\mathbf{y};\boldsymbol{\varphi_1}) + a_0\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) = 0$$
$$-(\mathbf{y};\boldsymbol{\varphi_2}) + a_0\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) = 0$$

$$a_0\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) = (\mathbf{y};\boldsymbol{\varphi_0})$$
$$a_0\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) = (\mathbf{y};\boldsymbol{\varphi_1})$$
$$a_0\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) + a_1\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_1}) + a_2\cdot(\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) = (\mathbf{y};\boldsymbol{\varphi_2})$$

$$\begin{pmatrix} (\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_2};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{y};\boldsymbol{\varphi_0}) \\ (\mathbf{y};\boldsymbol{\varphi_1}) \\ (\mathbf{y};\boldsymbol{\varphi_2}) \end{pmatrix} =$$

$$\begin{pmatrix} (\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) & \dots & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_m}) \\ (\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) & \dots & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_m}) \\ \dots & \dots & \dots & \dots \\ (\boldsymbol{\varphi_m};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_m};\boldsymbol{\varphi_1}) & \dots & (\boldsymbol{\varphi_m};\boldsymbol{\varphi_m}) \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_m \end{pmatrix} = \begin{pmatrix} (\mathbf{y};\boldsymbol{\varphi_0}) \\ (\mathbf{y};\boldsymbol{\varphi_1}) \\ \dots \\ (\mathbf{y};\boldsymbol{\varphi_m}) \end{pmatrix}$$

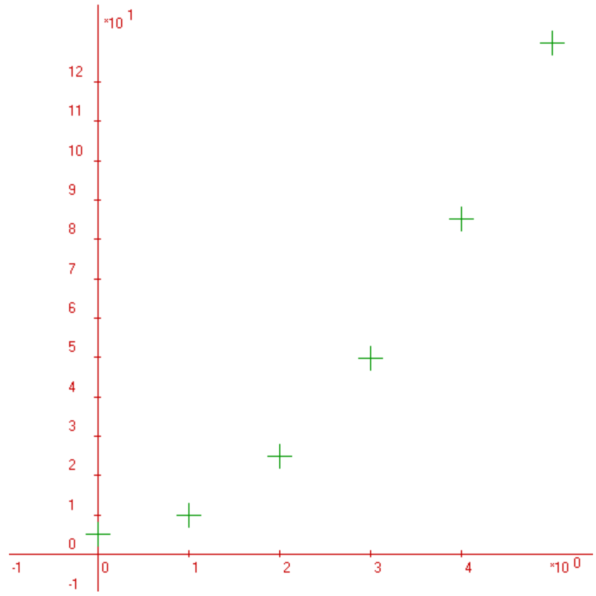
Aproximace funkce metodou nejmenších čtverců

Příklad: Při měření dráhy pohybujícího se tělesa byly v časových okamžicích t_i zjištěny následující hodnoty. Určete závislost dráhy na čase

i	0	1	2	3	4	5
t_i	0	1	2	3	4	5
s_i	5.2	10.1	24.9	50.0	85.2	130.0

$$s(t) = a_0 \cdot 1 \quad + a_1 \quad t \quad + a_2 \quad t^2$$
$$s(t) = a_0\varphi_0(t) + a_1\varphi_1(t) + a_2\varphi_2(t)$$

$$\begin{pmatrix} (\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_2};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{y};\boldsymbol{\varphi_0}) \\ (\mathbf{y};\boldsymbol{\varphi_1}) \\ (\mathbf{y};\boldsymbol{\varphi_2}) \end{pmatrix}$$



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Příklad: Při měření dráhy pohybujícího se tělesa byly v časových okamžicích t_i zjištěny následující hodnoty. Určete závislost dráhy na čase

i	0	1	2	3	4	5
t_i	0	1	2	3	4	5
s_i	5.2	10.1	24.9	50.0	85.2	130.0

$$s(t) = a_0 \cdot 1 \quad + a_1 \quad t \quad + a_2 \quad t^2$$
$$s(t) = a_0\varphi_0(t) + a_1\varphi_1(t) + a_2\varphi_2(t)$$

$$\begin{pmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 305.4 \\ 1200.7 \\ 5172.9 \end{pmatrix} \Longrightarrow \quad a_0 = 5.02; \qquad a_1 = -0.115; \qquad a_2 = 5.09$$

$$s = f(t) \approx 5.1 \cdot t^2 - 0.1 \cdot t + 5.0$$

$$(\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) = \sum \varphi_0(t_i)\varphi_0(t_i) = \sum 1 \cdot 1 = 6 \cdot 1 \cdot 1 = 6$$
$$(\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) = \sum \varphi_0(t_i)\varphi_1(t_i) = \sum 1 \cdot t_i = 0 + 1 + 2 + \cdots + 5 = 15 \qquad (\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) = (\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) = 15$$
$$(\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) = \sum \varphi_1(t_i)\varphi_1(t_i) = \sum t_i \cdot t_i = 0^2 + 1^2 + 2^2 + \cdots + 5^2 = 55$$
$$(\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) = \sum \varphi_0(t_i)\varphi_2(t_i) = \sum 1 \cdot t_i^2 = 0^2 + 1^2 + 2^2 + \cdots + 5^2 = 55 \qquad (\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) = (\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) = 55$$
$$(\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) = \sum \varphi_1(t_i)\varphi_2(t_i) = \sum t_i \cdot t_i^2 = 0^3 + 1^3 + 2^3 + \cdots + 5^3 = 225 \qquad (\boldsymbol{\varphi_2};\boldsymbol{\varphi_1}) = (\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) = 225$$
$$(\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) = \sum \varphi_2(t_i)\varphi_2(t_i) = \sum t_i^2 \cdot t_i^2 = 0^4 + 1^4 + 2^4 + \cdots + 5^4 = 979$$

$$(\mathbf{s};\boldsymbol{\varphi_0}) = \sum s_i\varphi_0(t_i) = \sum s_i \cdot 1 = 5.2 + 10.1 + 24.9 + 50.0 + 85.2 + 130.0 = 305.4$$
$$(\mathbf{s};\boldsymbol{\varphi_1}) = \sum s_i\varphi_1(t_i) = \sum s_i \cdot t_i = 5.2 \cdot 0 + 10.1 \cdot 1 + 24.9 \cdot 2 + \cdots + 130.0 \cdot 5 = 1200.7$$
$$(\mathbf{s};\boldsymbol{\varphi_2}) = \sum s_i(t_i)\varphi_2(t_i) = \sum s_i \cdot t_i^2 = 5.2 \cdot 0^2 + 10.1 \cdot 1^2 + 24.9 \cdot 2^2 + \cdots + 130.0 \cdot 5^2 = 5172.9$$