Solvability of Operator Equations

Survey Directed to Differential Equations

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Abstract

The contribution deals with solvability of operator equations of monotone type. The first part contains four abstract existence theorems for equations with operators being strongly monotone, monotone, weakly continuous or operators satisfying (M_0)-condition. The second part contains some examples of applications of abstract existence theorems to particular problems. The third part surveys auxiliary results that help verifying the assumptions.

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Introduction.

The subject of these lectures is the solvability of operator equations of type $A(u) = b$ on a Banach space V and its application to boundary value problems for differential equations.

The generalized formulation of many stationary boundary value problems for partial differential equations leads to operator equation of this type. Indeed, the generalized (the so-called weak) formulation consists in looking for an unknown function u from a Banach space V such that an integral identity containing u holds for each test function v from the space V. Since the identity is linear in v we can take its sides as values of continuous linear functionals at the element $v \in V$. Denoting the terms containing unknown u as the value of an operator A we obtain

$$\langle A(u), v\rangle = \langle b, v\rangle \quad \forall v \in V\,,$$

which is equivalent to equality of functionals on V, i. e. the equality of elements of V':

$$A(u) = b\,.$$

That is the reason why we are interested in solvability of operator equations.

Functional analysis yields tools for proving existence of generalized solutions to a relatively wide class of differential equations that appear in mathematical physics and industry. Moreover, often this functional analysis proposes an efficient numerical method for computing the solution.

Outline of the procedure.

The procedure of investigation of solvability to boundary value problems for differential equations proceeds in several steps:

- for a given differential equation we derive an integral identity. We multiply the equation by a test function, use Green's theorem (integration by parts) for lowering the order of derivatives and take into account the boundary conditions,
- we choose convenient function spaces for the unknown, for the test function and we formulate the problem,
- we check justification of the weak formulation, i. e. we prove that all terms in the integral identity define functionals and operators on the chosen spaces,

- we verify the assumptions of abstract existence theorem and thus prove that the problem admits a solution.

  This analysis is usually continued by

- proposing a numerical method for computing the solution,
- implementation of the method to computer, testing the method and finally
- computing the solution to the problem,

but that is another story.

Let us remark that besides the above mentioned weak formulation also the variational formulation is used. It consists in looking for a function u for which the integral functional (corresponding to the problem — the so-called energy functional) attains its minimum in the function space. Making the variation of the functional, i. e. the differentiating the functional in "admissible directions" v we obtain the weak formulation. Let us remark that not each weak formulation has corresponding variational formulation.

One-dimensional analog to existence theorems.

To obtain a first insight of abstract existence theorems we introduce a couple of trivial theorems.

In case the space V is a real line $\mathbb{R}$ (or its subset) the operator A is a function $f : \mathbb{R} \to \mathbb{R}$.

We start with the problem on a bounded interval, see Fig. 1:

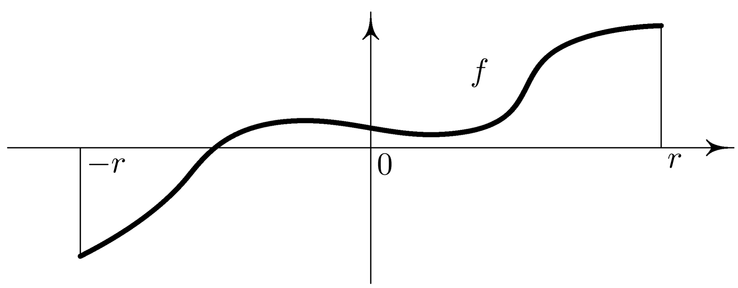


Fig. 1. Existence theorem on the segment.

Theorem A. *Let $f : [-r, r] \to \mathbb{R}$ be a continuous function satisfying*

$$f(-r) < 0 \,, \qquad f(r) > 0 \,.$$

Then there exists a point $x \in (-r, r)$ being the solution to the problem $f(x) = 0$.

The proof is simple, it can be carried out e. g. by the method of halving of the interval.

Further theorems will deal with the case of equations on the whole real line. They differ by the monotony condition, see Fig. 2. Each of the theorems has its infinite dimensional analog. To control the behaviour of the function f at infinity we introduce the notion of coercivity:

The function $f : \mathbb{R} \to \mathbb{R}$ is said to be **coercive** iff

$$\lim_{x \to -\infty} f(x) = -\infty \qquad \lim_{x \to \infty} f(x) = \infty \, .$$

The theorems deals with solvability of the equation $f(x) = y$.

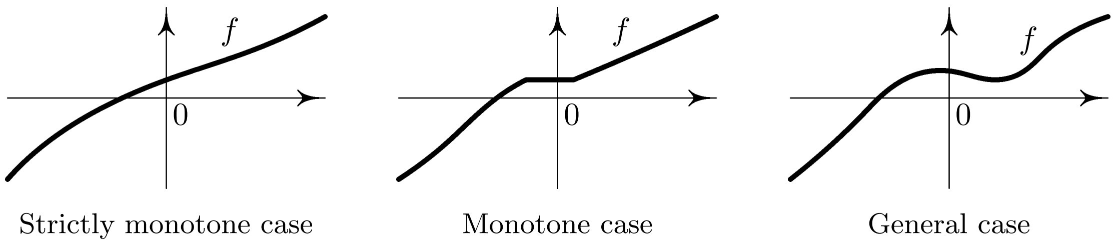


Fig. 2. Existence theorems on the real line.

Theorem B (strictly monotone case). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous coercive function which is increasing (strictly monotone).*

Then for each $y \in \mathbb{R}$ the equation admits unique solution.

Theorem C (monotone case). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous coercive function which is nondecreasing (monotone).*

Then for each $y \in \mathbb{R}$ the equation admits a solution. The set of solutions $\{x \in \mathbb{R} \mid f(x) = y\}$ forms a closed convex subset in $\mathbb{R}$. If the solution is unique then this set is single-point.

Theorem D (general case). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous coercive function.*

Then for each $y \in \mathbb{R}$ the equation admits a solution.

In the last theorem the solution need not be unique and the set of solutions may be disconnected.

Let us remark that in the one- and more-dimensional cases the existence of the solution is ensured by continuity and coercivity only, the monotony condition is redundant. In the infinite dimension case the situation is more complicated, the coercivity and continuity is not sufficient.

Part I

Abstract operator equations

We shall deal with the operator equation $A(u) = b$. We start with the finite-dimensional case, then we proceed to operator equations in Banach spaces.

1. Finite dimensional case.

The finite dimensional spaces can be identified with $\mathbb{R}^n$. Thus the operators on these spaces are real mappings $f : \mathbb{R}^n \to \mathbb{R}^n$. The scalar product and norm (absolute value) of elements (vectors) $x, y \in \mathbb{R}^n$ ($x \equiv (x^1, x^2, \ldots, x^n)$ and $y \equiv (y^1, y^2, \ldots, y^n)$) will be denoted by

$$(x, y) = \sum_{i=1}^{n} x^i y^i \,, \qquad |x| = \left[\sum_{i=1}^{n} (x^i)^2 \right]^{1/2} .$$

The distance between points $x, y \in \mathbb{R}^n$ is measured by $|x-y| = [(x-y, x-y)]^{1/2}$. Moreover, the distance induces in $\mathbb{R}^n$ natural convergence denoted by "$\to$":

$$x_n \to x \qquad \text{iff} \qquad |x_n - x| \to 0 \,.$$

Definitions.

We extend the properties of mappings from the real line $\mathbb{R}$ to $\mathbb{R}^n$. Let f be a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$.

Taking the definition of continuity based on the convergence the definition can remain without changes:

We say that f is **continuous** *iff*

$$x_n \to x \quad \Longrightarrow \quad f(x_n) \to f(x) \,.$$

The condition of the nondecreasing function

$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) \le f(x_2)$$

is equivalent to

$$(x_1 - x_2)(f(x_1) - f(x_2)) \ge 0 \,.$$

In the more dimensional case we just replace multiplication by the scalar product. Thus we obtain the definition of monotone function. Similarly we rewrite the definition of strictly monotone function:

We say that f is **monotone** *iff*

$$(f(x_1) - f(x_2), x_1 - x_2) \geq 0 \qquad \forall x_1, x_2 \in \mathbb{R}\,.$$

We say that f is **strictly monotone** *iff*

$$(f(x_1) - f(x_2), x_1 - x_2) > 0 \qquad \forall x_1, x_2 \in \mathbb{R} \quad x_1 \neq x_2\,.$$

In dimension one the coercivity condition consists of two limits. Using the sign function $\operatorname{sign} x = x/|x|$ these two limits can be written in one limit when $|x| \to \infty$. Replacing multiplication by the scalar product we obtain the definition in $\mathbb{R}^n$:

We say that f is **coercive** *iff*

$$\lim_{|x| \to \infty} \frac{(f(x), x)}{|x|} = \infty\,.$$

Existence theorem on the closed ball.

On the interval $[-r, r]$ of the real line we have a simple existence result, see Fig. 1, Theorem A. It can be extended to finite dimensional case.

We shall consider the problem on the closed ball

$$B_r = \{x \in \mathbb{R}^n \;\big|\; |x| \leq r\}\,.$$

In Theorem A if we draw the values of f by arrows in the space $\mathbb{R}$, we see that the condition $f(-r) < 0, f(r) > 0$ says that values of f at the end points $-r$ and r are directed outwards the segment $[-r, r]$.

This condition of outward oriented arrows $f(x)$ on the boundary can be extended to the ball B_r by means of scalar product $(f(x), x) > 0$. Indeed, by definition of the scalar product $(f(x), x) = |f(x)| \cdot |x| \cos\varphi$, we see that $(f(x), x)$ is positive iff the angle φ between vectors $f(x)$ and x is sharp, see Fig. 3. Thus we obtain the theorem:

Theorem 1.1. *Let $f : B_r \to \mathbb{R}^n$ be a continuous mapping satisfying on the boundary ∂B_r the condition of outward oriented values:*

$$(f(x), x) > 0 \quad \forall x\,, \; |x| = r\,. \tag{1.1}$$

Then there exists a point $x \in B_r$ being the solution to the problem $f(x) = 0$.

In the finite dimensional case the proof is not obvious although it is clear, that the continuous field of arrows directed outwards on the boundary must have a zero vector, see Fig. 3.

The mathematical proof is not constructive. It is based on the following Brouwer fixed point theorem:

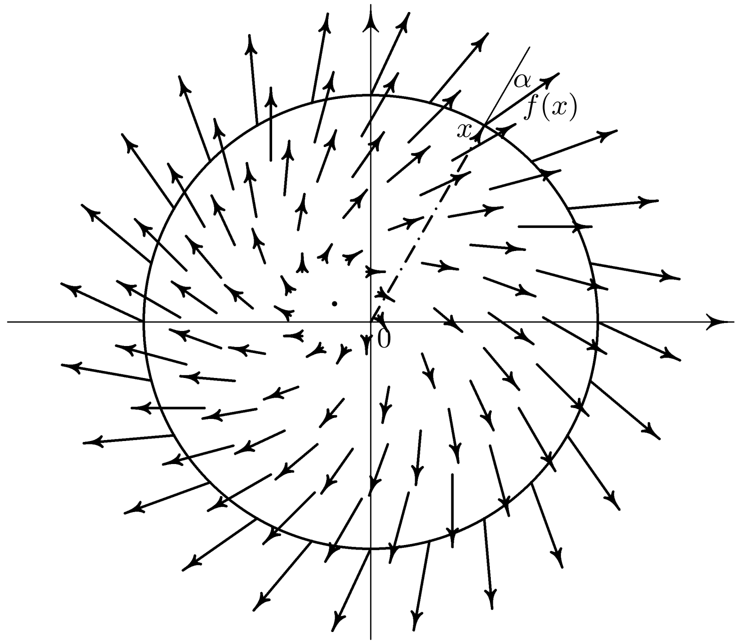


Fig. 3. Existence theorem for the ball B_r.

Theorem (Brouwer). *Let g be a continuous mapping from B_r into itself $g : B_r \to B_r$. Then the mapping has a fixed point, i. e. there exists $x \in B_r$ such that $g(x) = x$.*

The proof of this fundamental theorem is not simple, therefore we only refer e. g. to [4] for a proof via topological degree. The classical proof is based on homological algebra.

Proof of Theorem 1.1. We convert the problem of solvability of the equation into a fixed point problem. An element x is a solution of the equation $f(x) = 0$ iff x is a fixed point of the mapping g_ε $(\varepsilon > 0)$ defined by

$$g_\varepsilon(x) = x - \varepsilon f(x) .$$

The mapping is continuous. In order to be able to use the Brouwer theorem we find a constant $\varepsilon > 0$ such that g_ε maps B_r into itself.

The mapping f is continuous on the compact ball B_r, therefore it is bounded

$$|f(x)| \le L \quad \forall x \, , \, |x| \le r \, .$$

Moreover the condition (1.1) on the boundary ∂B_r for a continuous mapping on a compact implies inequality $(f(x), x) \ge K$ on ∂B_r, and further the same inequality with a smaller constant in a neighbourhood of B_r, i. e.

$$(f(x), x) \ge K/2 \quad \forall x \quad |x| \in (\rho, r] \, , \quad \rho < r \, .$$

We estimate $|g_\varepsilon(x)|^2$ in two cases: for $|x| \in [0,\rho]$ and for $|x| \in (\rho, r]$:

$$\begin{aligned} |g_\varepsilon(x)|^2 &= |x|^2 - 2\varepsilon(f(x),x) + \varepsilon^2 |f(x)|^2 \le \\ &\le \rho^2 + 2\varepsilon\rho L + \varepsilon^2 L^2 \qquad \text{in case} \quad |x| \in [0,\rho]\,, \\ &\le r^2 - 2\varepsilon K/2 + \varepsilon^2 L^2 \qquad \text{in case} \quad |x| \in (\rho, r]\,. \end{aligned}$$

There exists a small constant $\varepsilon > 0$ such that in both cases

$$|g_\varepsilon(x)| \le r \quad \forall x \in B_r\,,$$

i. e. the mapping g_ε maps B_r into itself and the Brouwer fixed point theorem yields the existence of a fixed point of g_ε. Thus the equation $f(x) = 0$ has a solution. □

Existence theorem in finite dimension.

The existence theorem on the ball implies the finite dimensional analog to Theorem D:

Theorem 1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous coercive mapping.*

Then it is surjective, i. e. the equation $f(x) = y$ has the solution for each $y \in \mathbb{R}^n$.

Proof. Let $y \in \mathbb{R}^n$. Let us consider the mapping $\widehat{f} : x \mapsto f(x) - y$. It is continuous, too. To apply Theorem 1.1 it remains to verify the condition (1.1) for $\widehat{f}$. The mapping f is coercive. Following the definition of the limit for each $K > 0$ there exists $r > 0$ such that $|x| \ge r$ implies $(f(x), x) > K\,|x|$. Choosing $K = |y|$ we obtain for $|x| = r$

$$(\widehat{f}(x), x) = (f(x) - y, x) \ge (f(x), x) - |y| \cdot |x| > 0$$

and the result follows. □

2. Infinite dimensional case — introduction.

Let us pass to infinite dimensional spaces. We shall deal with the Banach space V, i. e. the linear space with elements denoted by $u, v, \dots$ and equipped with a norm $\|\cdot\|$. The space is supposed to be complete with respect to this norm.

Continuous linear forms on V constitute the dual space denoted by V'. The value of the form $b \in V'$ at point $u \in V$ will be denoted by $\langle b, u \rangle$. The space V' is also a Banach space with the norm $\|f\| = \sup\{|\langle f, v\rangle| \;\big|\; v \in V,\ \|v\| \le 1\}\,.$

The special case of Banach spaces is the Hilbert space equipped with the scalar product (u, v) and the norm $\|u\| = \sqrt{(u, u)}$. By Riesz representation theorem the dual space V' can be identified with the space V and thus the duality map $\langle b, u\rangle$ can be identified with the scalar product (b, u).

Weak convergence.

Infinite dimensional spaces bring some difficulties. A closed bounded set, e. g. $B_r = \{u \in V \big| \; \|u\| \le r\}$ in not compact, which implies e. g. that a bounded sequence need not contain a convergent subsequence. Thus continuous mapping on B_r need not be bounded, bounded continuous mapping on B_r need not have its maximum, etc.

This is why we introduce the following concept: Besides the strong convergence on the Banach space V denoted by "$\to$"

$$u_n \to u \qquad \text{iff} \qquad \|u_n - u\| \to 0$$

we introduce the weak convergence on V denoted by a halfarrow "$\overset{w}{\rightharpoonup}$"

$$u_n \overset{w}{\rightharpoonup} u \qquad \text{iff} \qquad \langle b, u_n - u\rangle \to 0 \qquad \forall\, b \in V' .$$

Clearly, the weak convergence is weaker, i. e. each strongly converging sequence is also weakly convergent, but the converse is not true.

Similarly, on the dual space V' we have the strong and weak convergences:

$$\begin{aligned} b_n &\to b \qquad \text{iff} \qquad \|b_n - b\|_{V'} \to 0 \,, \\ b_n &\overset{w}{\rightharpoonup} b \qquad \text{iff} \qquad \langle \varphi, b_n - b\rangle \to 0 \qquad \forall\, \varphi \in V'' \,, \end{aligned}$$

where V'' is the second dual space, i. e. the space of linear continuous functionals on V'. We can get some elements of V'' if we assign to any $u \in V$ a functional $\varphi \in V''$ by the relation $\langle \varphi, b\rangle = \langle b, u\rangle$, but in general we do not obtain the whole space V''.

Reflexive spaces.

The spaces in which V can be identified with V'' by the above mentioned canonical imbedding are called **reflexive**. In these spaces the weak convergence on V' can be defined as

$$b_n \overset{w}{\rightharpoonup} b \qquad \text{iff} \qquad \langle b_n - b, v\rangle \to 0 \quad \forall\, v \in V \;\; .$$

The main contribution of the weak convergence is the fact, that it makes the closed balls B_r of infinite dimensional reflexive spaces compact:

Theorem 2.1. *Let V be a reflexive Banach space. Then the closed ball $B_r = \{u \in V \,\big|\, \|u\| \le r\}$ is weakly sequentially compact, i.e. each bounded sequence $\{u_n\}$ contains a subsequence $\{u_{n'}\}$ weakly converging to an element $u \in B_r$:* $u_{n'} \overset{w}{\rightharpoonup} u$.

The theorem is a consequence of the Eberlein-Schmulian theorem, which moreover asserts that if the ball B_r is weakly sequentially compact then the Banach space is reflexive, see [5], [8], [11].

Let us remark that Hilbert spaces are reflexive. In finite dimensional spaces both the strong and weak convergences coincide.

Abstract operator equation.

We shall consider an operator $A : V \to V'$. The equation

$$A(u) = b$$

for $b \in V'$ means the equality of functionals on V, i.e. the problem reads:

$$\text{(P)} \qquad \text{Find } u \in V \text{ such that } \quad \langle A(u), v\rangle = \langle b, v\rangle \quad \forall v \in V\,.$$

The problem (P) is an abstract formulation of many problems, namely boundary value problems for ordinary differential equations and stationary partial differential equations — see Part II.

Definitions.

Replacing the scalar product (y, x) by the duality map $\langle f, u\rangle$ we rewrite the definitions for operators on Banach spaces.

Let V be a Banach space with its dual V' and A an operator $A : V \to V'$. We say that the operator A is:

— **coercive** *iff*

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = \infty \tag{2.1}$$

— **monotone** *iff*

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \ge 0 \qquad \forall\, u_1, u_2 \in V\,, \tag{2.2}$$

— **strictly monotone** *iff*

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle > 0 \qquad \forall\, u_1, u_2 \in V\,,\ u_1 \ne u_2\,, \tag{2.3}$$

— **bounded** *iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $M > 0$ (M depending on r) such that*

$$\|u\| \le r \quad \Longrightarrow \quad \|A(u)\| \le M \quad \forall\, u \in V\,, \tag{2.4}$$

— **continuous** *iff it maps convergent sequences into convergent ones i. e.*

$$u_k \to u \quad \Longrightarrow \quad A(u_k) \to A(u) \quad \forall\, u_k, u \in V\,.$$

We add four new notions: *The operator A is*
— **strongly monotone** *iff there exists $\alpha > 0$ such that*

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \alpha\, \|u_1 - u_2\|^2 \qquad \forall\, u_1, u_2 \in V\,,$$

— **continuous on finite-dimensional subspaces** *iff for each finite dimensional subspace V_n the restriction of the operator to V_n (more precisely the "Galerkin approximation" of A) is continuous i. e.*

$$\{u_k\} \subset V_n \quad u_k \to u \quad \Longrightarrow \quad A(u_k)\big|_{V_n} \to A(u)\big|_{V_n}\,, \tag{2.5}$$

— **weakly continuous** *iff it maps weakly converging sequences into weakly converging ones i. e.*

$$u_k \overset{w}{\rightharpoonup} u \quad \Longrightarrow \quad A(u_k) \overset{w}{\rightharpoonup} A(u) \qquad \forall\, u_k, u \in V\,. \tag{2.6}$$

— **hemicontinuous** *(weakly continuous on lines) iff it is weakly continuous on lines i. e.*

$$t_k \to 0 \quad \Longrightarrow \quad A(u + t_k v) \overset{w}{\rightharpoonup} A(u) \quad \forall\, u, v \in V \qquad t_k \in \mathbb{R}\,. \tag{2.7}$$

Remark.

In general there is no relation between continuity and weak continuity. There are another two continuities:

A is *strongly continuous* iff it maps weakly convergent sequences into strongly convergent ones, i. e. $u_k \overset{w}{\rightharpoonup} u \quad \Longrightarrow \quad A(u_k) \to A(u)\,,$

A is *demicontinuous* iff it maps strongly convergent sequences into weakly convergent ones, i. e. $u_k \to u \quad \Longrightarrow \quad A(u_k) \overset{w}{\rightharpoonup} A(u)\,.$

The relations between different continuities are surveyed in Section 14.

Uniqueness of the solution.

The strict monotony ensures uniqueness of the solution:

Theorem 2.2. *Let the operator A be strictly monotone, see (2.3). Then equation $A(u) = b$ cannot have two different solutions.*

Proof. Supposing the equation has two solutions $u_1, u_2 \in V$, we obtain $A(u_1) = A(u_2)$ and thus $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle = 0$. Due to condition (2.3) we conclude $u_1 = u_2$. □

3. Strongly monotone operators.

In this section we prove an analog to Theorem B. The strong monotony

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2 \quad \forall\, u_1, u_2 \in V \ , \qquad (\alpha > 0)$$

is a stronger condition than strict monotony (2.3). It means that the real function f is not only increasing but its slope is bounded from bellow. More precisely if f is differentiable, then strong monotony means that $f' \geq \alpha > 0$.

The strongly monotone operators forming a special subclass of monotone operators are in a certain sense close to linear elliptic operators. Existence and uniqueness of solutions can be easily proved by the Banach fixed point theorem.

In this section we restrict ourselves to the case of V being a Hilbert space. Thus we can identify the functionals of V' with the elements of V and replace the duality map $\langle b, v \rangle$ by the scalar product (b, v).

Theorem 3.1. *Let V be a Hilbert space and $A : V \to V$ an operator which is*
— *strongly monotone, i. e. there exists $\alpha > 0$ such that*

$$(A(u_1) - A(u_2), u_1 - u_2) \geq \alpha \, \|u_1 - u_2\|^2 \qquad \forall\, u_1, u_2 \in V \ , \tag{3.1}$$

— *and Lipschitz continuous, i.e. there exists $M > 0$ such that*

$$\|A(u_1) - A(u_2)\| \ \leq \ M \, \|u_1 - u_2\| \qquad \forall\, u_1, u_2 \in V \, . \tag{3.2}$$

Then the equation $A(u) = b$ admits a unique solution for each $b \in V$.

Proof. Again we convert the problem of solving equation $A(u) = b$ into a fixed point problem. An element u is a solution to the equation iff u is a fixed point of the mapping

$$T_\varepsilon(u) = u - \varepsilon(A(u) - b)$$

for a constant $\varepsilon > 0$.

We find $\varepsilon > 0$ such that the mapping T_ε is contractive, i. e.

$$\|T_\varepsilon(u_1) - T_\varepsilon(u_2)\| \leq c \cdot \|u_1 - u_2\|$$

with a constant $c < 1$. In the estimate we use the inequalities (3.1), (3.2):

$$\|T_\varepsilon(u_1) - T_\varepsilon(u_2)\|^2 = \|(u_1 - u_2) - \varepsilon(A(u_1) - A(u_2)\|^2 =$$

$$= \|u_1 - u_2\|^2 - 2\varepsilon(A(u_1) - A(u_2), u_1 - u_2) + \varepsilon^2 \|A(u_1) - A(u_2)\|^2 \leq$$

$$\leq \|u_1 - u_2\|^2 (1 - 2\varepsilon\alpha + \varepsilon^2 M^2) \, .$$

Function $\varphi(\varepsilon) = 1 - 2\varepsilon\alpha + \varepsilon^2 M^2$ is a parabola open upwards passing through the point $(0,1)$. It attains its minimum $1 - \alpha^2/M^2 < 1$ at $\varepsilon = \alpha/M^2$.

Thus for $\varepsilon = \alpha/M^2$ the mapping T_ε is contractive with constant less than one:
$c = (1 - \alpha^2/M^2)^{1/2} < 1$. Following the Banach theorem the contractive mapping T_ε on the complete metric space V admits unique fixed point u which is unique solution to the equation. □

Remarks.

(a) If A is a linear operator on a Hilbert space, the condition (3.1) is equivalent to *ellipticity* condition

$$(A(u), u) \geq \alpha \cdot \|u\|^2 \quad \forall u \in V\,.$$

Further, for linear operators the conditions of Lipschitz continuity, continuity and boundedness are equivalent and (3.2) can be replaced by

$$\|A(u)\| \leq M\,\|u\|\,.$$

Thus in case of linear operator Theorem 3.1 is identical to the well known Lax-Milgram lemma.

(b) The proof of Theorem 3.1 by means of the Banach fixed point theorem is constructive and yields an important approximate method. The sequence of approximate solutions $\{u_k\}$ defined by

$$u_0 \in V \text{ — arbitrary,} \qquad u_{k+1} = T_\varepsilon(u_k)\,, \quad k = 0, 1, 2, \ldots \tag{3.3}$$

converges in the norm to the solution u of the equation. We can even estimate the speed of convergence. Summing the inequalities

$$\|u_{j+1} - u_j\| \;\leq\; c^j\,\|T_\varepsilon(u_0) - u_0\|$$

(where $c < 1$ is the contraction constant) for $j = k, k+1, \ldots$ and using triangle inequality we obtain

$$\|u - u_k\| \;\leq\; \frac{c^k}{1-c}\,\|T_\varepsilon(u_0) - u_0\|\,. \tag{3.4}$$

4. Galerkin approximation.

The proofs of further three abstract existence theorems are based on the notion of Galerkin approximation. Since we cannot treat the equation on the infinite dimensional space directly, we construct a sequence of finite dimensional problems with a sequence of corresponding solutions u_n.

Let us consider Banach space V, operator $A : V \to V'$ and $b \in V'$. The equation $A(u) = b$ on the infinite-dimensional space V means the problem:

$$\text{(P)} \qquad \text{Find } u \in V \text{ such that} \quad \langle A(u), v \rangle = \langle b, v \rangle \quad \forall v \in V\,.$$

The problem (P) can be restricted to problem (P_n) on a finite-dimensional subspace $V_n \subset V$ as follows:

$$(\mathrm{P}_n) \qquad \text{Find } u_n \in V_n \text{ such that} \quad \langle A(u_n), v \rangle = \langle b, v \rangle \quad \forall v \ \in V_n\,.$$

The problem (P_n) is called **Galerkin approximation** of the problem (P).

Solvability of the problem (P_n).

We shall need to know solvability of the problem (P_n):

Lemma 4.1. *Let V be a Banach space and $A : V \to V'$ a coercive operator continuous on finite-dimensional subspaces. Let V_n be a finite-dimensional subspace.*

Then the Galerkin approximation (P_n) of the problem (P) admits its solution u_n. In addition, the solutions to problems (P_n) and (P) are bounded

$$\|u_n\| \le r\,, \tag{4.1}$$

where the constant r is independent of the choice of space V_n.

Proof. The Galerkin approximation is equivalent to an equation $f(x) = y$ on $\mathbb{R}^n$. Indeed, let V_n be an n-dimensional subspace with base $(w_1, w_2, w_3, \dots, w_n)$. Any element $u_n \in V_n$, $u_n = \sum_k x_k w_k$ can be represented by the vector of its coordinates $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with respect to the base. Similarly, any functional on V_n $b|_{V_n} = b_n \in V_n'$ can be represented by the vector $y = (y_1, \dots, y_n)$, where $y_k = \langle b, w_k \rangle$. In this way we assign to the operator A a vector function $f : \mathbb{R}^n \to \mathbb{R}^n$

$$f : x \equiv (x_1, \dots, x_n) \in \mathbb{R}^n \quad \mapsto \quad \left\{ \left\langle A\left(\sum_{i=1}^{n} x_i w_i \right), w_k \right\rangle \right\}_{k=1}^{n} \in \mathbb{R}^n$$

and obtain the equation $f(x) = y$ on $\mathbb{R}^n$.

Since A is coercive and continuous on finite dimensional subspaces, the function f is coercive and continuous on $\mathbb{R}^n$. Thus Theorem 1.2 yields the existence of the solution x and successively the existence of the solution u_n.

The estimate is a consequence of the coercivity. Indeed, limit (2.1) means that for each $K > 0$ there exists $r > 0$ (r depending on K) such that

$$\|u\| > r \quad \Longrightarrow \quad \langle A(u), u \rangle > K \cdot \|u\|\,.$$

Transposing the last implication we obtain

$$\langle A(u), u\rangle \leq K \cdot \|u\| \quad \Longrightarrow \quad \|u\| \leq r\,.$$

For $K = \|b\|$ the equality in (P_n) gives

$$\langle A(u_n), u_n\rangle = \langle b, u_n\rangle \leq K \cdot \|u_n\|$$

and the implication yields the desired estimate $\|u_n\| \leq r$, where r depends only on $K = \|b\|$. □

Approximative property.

Existence of the solution u will be proved by means of a sequence of solutions u_n to problems (P_n) corresponding to a sequence of finite dimensional subspaces V_n of the space V. Clearly, not all sequences $\{V_n\}$ can be used. Any element $v \in V$ should be "reachable" as a limit of elements of these subspaces.

Definition. *We say that a sequence $\{V_n\}$ of subspaces of V has* **approximative property** *iff*

$$\forall v \in V \quad \exists \{v_n\}_n\,,\ v_n \in V_n \quad \text{such that} \quad v_n \to v\ . \tag{4.2}$$

Another problem is finding such a sequence of subspaces. In case of separable space V we have:

Lemma 4.2. *Let V be a separable Banach space. Then there exists a sequence of finite dimensional subspaces $\{V_n\}$ with approximative property.*

Proof. The separable space V contains a countable sequence $\{w_1, w_2, w_3, \ldots\}$ dense in V. Its first n members generate a finite dimensional subspace V_n. Since $\{w_n\}$ was dense, any element of V is the limit of elements of $\{V_n\}$. Clearly, in the sequence $\{w_n\}$ we can omit members which are dependent on preceding members. □

Remarks.

(a) If the space V is not separable we have an incountable system of finite dimensional subspaces and we must work with generalized sequences called *nets*.

(b) Thanks to Lemma 4.1 and Lemma 4.2 we have a bounded sequence $\{u_n\}$ of solutions to problems (P_n). Due to Theorem 2.1 the sequence $\{u_n\}$ contains a subsequence $\{u_{n'}\}$ weakly converging to an element u. But u need not be the solution of problem (P). It must be ensured by another assumptions:

in Theorem 5.1 it is ensured by monotony, in Theorem 6.1 by weak continuity and in Theorem 7.1 by (M_0) condition.

5. Monotone operators.

The infinite dimensional analog of Theorem C is the following theorem:

Theorem 5.1 (Minty-Browder). *Let V be a reflexive Banach space and $A : V \to V'$ an operator which is*
— coercive (2.1),
— continuous on finite-dimensional subspaces (2.5),
— monotone (2.2).

Then the equation $A(u) = b$ has a solution for each $b \in V'$.

Moreover, the solutions form a closed convex set, i.e. for each $b \in V'$ the set $A^{-1}(b) = \{u \in V \mid A(u) = b\}$ is closed and convex.

The proof is based on the reformulation of the equation by means of the so-called Minty trick [7]:

Lemma 5.2 (Minty). *Let A be a monotone (2.2) hemicontinuous (2.7) operator and $u \in V$. Then the following two conditions are equivalent:*

$$\langle A(u), v-u\rangle \geq 0 \qquad \forall v \in V\,, \tag{5.1}$$

$$\langle A(v), v-u\rangle \geq 0 \qquad \forall v \in V\,. \tag{5.2}$$

Remark. Lemma holds even if we replace the space V by a closed convex subset $K \subset V$, see [7].

We can apply the lemma to monotone operators since assumed continuity on finite dimensional subspaces implies hemicontinuity.

Proof. Monotony condition written for v and u yields

$$\langle A(v), v-u\rangle \geq \langle A(u), v-u\rangle\,.$$

Thus (5.1) implies (5.2).

The opposite implication is proved by means of hemicontinuity. Let us assume (5.2). Let $w \in V$. Inserting $v = u + t(w-u)\,,\ t > 0$ in (5.2) we obtain

$$\langle A(u+t(w-u)), t(w-u)\rangle = t\langle A(u+t(w-u)), w-u\rangle \geq 0\,.$$

After dividing the inequality by $t > 0$ we pass to the limit $t \to 0_+$. Hemicontinuity implies $A(u+t(w-u)) \overset{w}{\rightharpoonup} A(u)$ and the inequality (5.1) follows. □

Existence of the solution will be proved by the following topological theorem:

Theorem on nonempty intersection.. *Let $\{U_\iota\,, \iota \in I\}$ be an arbitrary system of closed subsets of a compact topological space such that any intersection of a finite number of U_ι is nonempty.*

Then also the infinite intersection $\cap\{U_\iota\,, \iota \in I\}$ is nonempty.

Proof of Theorem 5.1. It is sufficient to prove the existence for the equation $A(u) = 0$. Indeed, for any $b \in V'$ the operator $\widehat{A}(u) = A(u) - b$ is again coercive, continuous on finite dimensional subspaces and monotone.

Let us realize that for $u \in V$ the condition

$$\langle A(u), v\rangle = 0 \qquad \forall v \in V \tag{5.3}$$

is equivalent to (5.1). Equality (5.3) implies inequality (5.1). The converse implication is also true. Indeed, for arbitrary $v \in V$ inserting $2u - v$ and 0 for v into (5.1) we obtain two opposite inequalities and (5.3) follows.

Using the preceding argument and Lemma 5.2 we see that $u \in V$ is the solution to our problem (P) iff (5.2) holds. For $v \in V$ let us denote

$$U(v) = \{u \in V\,,\ \langle A(v), v - u\rangle \geq 0\}\,.$$

Then the intersection $U_V = \cap\{U(v)\,, v \in V\}$ is the set of solutions and we have to prove that it is nonempty.

Let us verify the assumptions of Theorem on nonempty intersection. The space V is not compact. Due to Lemma 4.1 the solutions are in the bounded ball B_r. Thus we can restrict V to the closed ball B_r. With the weak topology the ball B_r forms a compact topological space.

For each $v \in V$ the set $U(v)$ is a closed halfspace. We put $\widehat{U}(v) = U(v) \cap B_r$. Since closed convex subsets are also closed in the weak topology, $\{\widehat{U}(v)\,, v \in V\}$ is a system of closed subsets of the compact topological space B_r.

It remains to verify that finite intersections are nonempty. Let $v_1, v_2, \ldots, v_n$ be arbitrary elements of V. Let us denote by V_n the finite dimensional space generated by $v_1, \ldots, v_n$. Using Lemma 5.2 we see that $u_n \in (U(v_1) \cap \ldots \cap U(v_n))$ is equivalent to the statement that u_n is a solution to the problem (P_n) on the finite dimensional subspace V_n. Due to Lemma 4.1 the solution u_n exists and is in B_r. Thus the finite intersection $\widehat{U}(v_1) \cap \cdots \cap \widehat{U}(v_n)$ is nonempty since it contains at least u_n.

Thus following Theorem on the nonempty intersection we can conclude that also the infinite intersection U_V is nonempty and thus the solution exists. Moreover, the set of solution U_V is closed and convex since it is an intersection of closed convex sets. □

Let us remark that in Theorem 5.1 the space V need not be separable.

6. Weakly continuous operators.

The infinite dimensional analog to Theorem D is the following theorem for weakly continuous operators:

Theorem 6.1. *Let V be a separable reflexive Banach space and $A : V \to V'$ be an operator which is*
— coercive (2.1),
— weakly continuous (2.6).

Then the equation $A(u) = b$ has a solution for each $b \in V'$.

Proof. Let V_n be a sequence of finite dimensional subspaces with the approximative property of Lemma 4.2. Due to Lemma 4.1 the corresponding sequence of the problems (P_n) yields the sequence of solutions u_n. The solutions are bounded due to estimate (4.1). Since V is reflexive the sequence $\{u_n\}$ contains a subsequence $u_{n'}$ weakly converging to an $u \in B_r$, see Theorem 2.1.

It remains to prove that the limit u is the solution. Due to the weak continuity of the operator A we have also $A(u_{n'}) \stackrel{w}{\rightharpoonup} A(u)$. Let $v \in V$ be arbitrary. We cannot insert it into the equality in (P_n) since $v \notin V_n$. Let $\{v_n\}$ be an approximating sequence ensured by (4.2), $v_n \in V_n$ and $v_n \to v$. It can be inserted into the equality

$$\langle A(u_n), v_n \rangle = \langle b, v_n \rangle .$$

Let us consider the subsequence $\{n'\}$ and let us pass to the limit. On the right-hand side we obtain $\langle b, v \rangle$. The left-hand side decomposed into two parts

$$\langle A(u_{n'}), v_{n'} \rangle = \langle A(u_{n'}), v_{n'} - v \rangle + \langle A(u_{n'}), v \rangle \tag{6.1}$$

tends to $\langle A(u), v \rangle$. Indeed, the first term tends to zero since

$$|\langle A(u_{n'}), v_{n'} - v \rangle| \le \|A(u_{n'})\| \cdot \|v_{n'} - v\|$$

and $A(u_{n'})$ is bounded. The second term converges due to $A(u_{n'}) \stackrel{w}{\rightharpoonup} A(u)$.

Thus we have obtained $\langle A(u), v \rangle = \langle b, v \rangle$. Since v was arbitrary the limit u is the solution and the proof is complete. □

7. Operators satisfying the (M_0)-condition.

The last abstract existence theorem is the most general. It is another infinite dimensional analog to Theorem D.

Theorem 7.1. *Let V be a separable reflexive Banach space and $A : V \to V'$ an operator which is*

— *coercive (2.1),*
— *bounded (2.4),*
— *continuous on finite-dimensional subspaces (2.5) and*
— *satisfying the so-called (M_0)-condition:*

$$(M_0) \qquad u_n \overset{w}{\rightharpoonup} u\,, \quad A\,u_n \overset{w}{\rightharpoonup} b\,, \quad \langle A\,u_n, u_n\rangle \to \langle b, u\rangle \quad \Longrightarrow \quad A(u) = b\,.$$

Then the equation $A(u) = b$ admits a solution for each $b \in V'$.

Proof. In the beginning the proof follows the proof of Theorem 6.1. Let $\{V_n\}$ be the sequence of finite dimensional subspaces and $\{u_n\}$ the corresponding sequence of solutions u_n to problems (P_n). Due to estimate (4.1) the sequence $\{u_n\}$ is bounded. The sequence $\{A(u_n)\}$ is bounded, too, since the operator A is bounded. Both V and V' are reflexive thus by double using of Theorem 2.1 we obtain a subsequence $\{n'\}$ such that

$$u_{n'} \overset{w}{\rightharpoonup} u \qquad \text{and} \qquad A(u_{n'}) \overset{w}{\rightharpoonup} f\,,$$

where u, f are elements $u \in V$, $f \in V'$.

First, we prove that $f = b$. Let $v \in V$. Let $\{v_n\}$ be a sequence given by the approximative property (4.2). We insert v_n into the equality (P_n) and restrict ourselves to the subsequence

$$\langle A(u_{n'}), v_{n'}\rangle = \langle b, v_{n'}\rangle\,.$$

Let us pass to the limit. The right-hand side tends to $\langle b, v\rangle$, the left hand-side converges to $\langle f, v\rangle$ due to $A(u_{n'}) \overset{w}{\rightharpoonup} f$ using the same argument as in (6.1). Thus $\langle f, v\rangle = \langle b, v\rangle$. Since v was arbitrary we proved $f = b$.

The final step will be accomplished by using $(M)_0$-condition for subsequence $\{n'\}$. It remains to verify its last assumption

$$\langle A(u_{n'}), u_{n'}\rangle \to \langle b, u\rangle\,.$$

It follows from the identity (P_n) which implies

$$\langle A(u_{n'}), u_{n'}\rangle = \langle b, u_{n'}\rangle \to \langle b, u\rangle\,.$$

The assertion of (M_0)-condition yields $A(u) = b$ and the proof is complete. □

8. Convergence of Galerkin approximations.

In the applications we are able to compute numerically only the approximative solution u_n. Thus we are interested in the rate of the convergence.

In case of the strongly monotone operator (Theorem 3.1), we have strong convergence of approximative solutions u_n constructed by the contractive operator T_ε, see Section 3, Remark (b). Moreover, (3.4) yields an error estimate.

In the other cases we used Galerkin approximations. In general we do not have convergence of the solutions u_n. The proofs yield the weak convergence of a subsequence $\{u_{n'}\}$ only. If the problem (P) has two different solutions, we cannot expect more. Indeed, some terms of the sequence $\{u_n\}$ can converge to one solution, the other to another solution. If the solution unique, then the whole sequence converges:

Theorem 8.1. *Let the assumptions of Theorem 6.1 or 7.1 be satisfied and let the problem (P) have at most one solution.*

Then the whole sequence of solutions $\{u_n\}$ converges weakly to the solution u.

Proof. The proof is a consequence the following general property of the convergence: *Let a sequence in a compact space be not convergent. Then it contains at least two subsequences converging to different limits.*

Our sequence of solutions u_n is contained in the ball B_r, which is compact with respect to weak convergence. Therefore, if the solution u is unique, then the whole sequence is converging. □

The theorems ensured weak convergence of approximative solutions only. If we add another assumption we obtain strong convergence:

Theorem 8.2. *Let the assumptions of Theorem 6.1 or 7.1 be satisfied. Let the operator A satisfy the so called* **condition (S_0)**

$$\text{(8.1)} \qquad \Big[u_n \overset{w}{\rightharpoonup} u\,,\ A(u_n) \overset{w}{\rightharpoonup} b\,,\ \langle A(u_n), u_n\rangle \to \langle b, u\rangle \Big] \quad \Longrightarrow \quad u_n \to u\ .$$

Let $\{u_n\}$ be the sequence of approximative solutions.

Then if a subsequence $\{u_{n'}\}$ converges weakly, then it converges also strongly to the solution u.

Proof. Any subsequence of weakly converging approximate solutions $u_{n'}$ satisfies the assumption of (S_0) condition. Indeed, $u_n \overset{w}{\rightharpoonup} u$, $A(u_n) \overset{w}{\rightharpoonup} b$, and $\langle A(u_n), u_n\rangle = \langle b, u_n\rangle \to \langle b, u\rangle$ thus the strong convergence to the solution follows and the proof is complete. □

Part II
Application to differential equations

In the second part we show application of the abstract existence theorems to boundary value problems for ordinary and partial differential equations. We start with brief characterization of individual steps, then we show the application on the examples.

9. The procedure in general.

Let Ω be a bounded domain with Lipschitz boundary Γ. We shall consider a boundary value problem for a partial differential equation of second order in divergent form on Ω with suitable boundary conditions on Γ. We shall look for the weak solution in a convenient function space V.

Integral identity.

The basis for weak or operator formulation is the integral identity. We multiply the differential equation by the so-called test function v with suitable boundary conditions and integrate the equality over the domain Ω. Using integration by parts (see Part III) we lower the highest derivatives of unknown function; we pass them to the test function. Dealing with boundary integrals we use boundary conditions.

Boundary conditions are of two kinds: stable and unstable. In case of a second order equation the former consist in prescribing values of the unknovn u, the latter in prescribing first order derivatives of the unknown. The stable boundary conditions are built in the formulation explicitly by the choice of the space. The unstable boundary conditions are included in the integral identity. The nonhomogeneous unstable conditions yield the boundary integrals.

Function space.

According to integral identities we choose convenient function spaces. The basic space is usually Sobolev space — a space of functions having generalized integrable derivatives, see Section 12. The solution will be looked for in a narrower space V — a subspace of the basic Sobolev space of functions satisfying stable boundary conditions in a generalized sense — sense of traces, see [9].

If the stable boundary conditions are non-zero, then they do not form a function space. We choose a function u_0 from the basic Sobolev space which satisfies the prescribed boundary conditions. Then we look for the solution in the space $V + u_0$ shifted by the function u_0. Another way consists in replacing the solution u with $u_0 + u^*$. Then the new unknown u^* already satisfies homogeneous boundary conditions.

Weak formulation.

Choosing a convenient basic Sobolev space W and its subspace V satisfying homogeneous boundary condition, we obtain the weak formulation:

Find $u \in W$, such that $u - u_0 \in V$ and the integral identity holds for all test functions $v \in V$.

Replacing the solution u by $u + u_0$ the weak formulation reads:

Find $u + u_0$, such that $u \in V$ and the integral identity (with u replaced by $u + u_0$) holds for all test functions $v \in V$.

Remark. Let us underline that in the weak formulation of the second order differential equation instead of looking for a twice continuously differentiable solution we look for the solution having integrable first order derivatives which even need not be continuous. Instead of pointwise equality (equation satisfied in all points of the domain Ω) we require that the integral identity is satisfied for all test functions.

Justification of the weak formulation.

Before further analysis we must justify the formulation, i. e. to prove that under the specified assumption for data, the unknown u and test function v the integrals in the integral identity make sense and are finite.

Each term of the integral identity is estimated by means of Cauchy or Hölder inequality. In case of nonlinear equations the Theorem on Nemytskij operators (Section 13) yields the desired result.

Operator formulation.

Since the integral identity is linear in the test function $v \in V$, the terms containing the unknown u define an operator $A : V \to V'$. Indeed, they define the value $\langle A(u), v\rangle$ of functional $A(u)$ at the test function v. The remaining terms define the right hand side functional $b \in V'$.

Applications of monotone operator theory.

The next step consists in verifying the assumptions. It is convenient to divide the operator into several parts. As a byproduct of Theorem on Nemytskij operator we obtain continuity and boundedness of the operator. Coercivity is ensured by an growth estimate from below

$$\langle A(u), u\rangle \geq \alpha\|u\|^2 \qquad (\alpha > 0)$$

for the principal part. The remaining parts must not violate it.

In the nonlinear case the operator usually is not monotone. Quasilinear operators use to be weakly continuous. The differential operators with strictly monotone principal part use to satisfy condition (M_0). If the coefficients are differentiable, the conditions can be formulated in terms of derivatives.

10. Examples.

Example 1. A simple ordinary differential equation.

We shall deal with the boundary value problem for a second order differential equation with homogeneous boundary condition

$$\begin{aligned} &-u'' + g(u) = f \quad \text{in } I = (0,1) \\ &u(0) = u(1) = 0\,, \end{aligned} \tag{10.1}$$

where f is a given function, $f \in L^2(I)$. We shall investigate four cases of the function g:

(a) $g(\xi) = c \cdot \xi\,, \quad c > 0\,,$

(b) $g(\xi) = \xi^3$

(c) $g(\xi)$ is a continuous nondecreasing function,

(d) $g(\xi)$ is an arbitrary continuous function.

In order to be able to use the monotone operator theory we reformulate the problem into the form of operator equation on a Banach space V.

Integral identity.

We multiply the equation by a function v and integrate the it with respect to x over I. Due to boundary condition $u(0) = u(1) = 0$ we choose the same condition for test function v. Integrating the first term by parts and using the condition $v(0) = v(1) = 0$ we obtain the integral identity

$$\int_I u'v'\,\mathrm{d}x + \int_I g(u)v\,\mathrm{d}x = \int_I fv\,\mathrm{d}x\,. \tag{10.2}$$

Function space.

The first term on the left-hand side is a bilinear form with first order derivatives. It is the principal part of the scalar product (12.9) of the Sobolev space $W^{1,2}(I)$. Indeed, with $v = u$ we obtain the principal part of the norm (12.8)

$$\|u\|_{1,2}^2 = \int_I \left(|u'(x)|^2 + |u(x)|^2\right) \mathrm{d}x \,. \tag{10.3}$$

Thus Sobolev space $W^{1,2}(I)$ (often denoted also by $H^{1,2}(I)$ or $H^1(I)$) is convenient for the basic space. Taking boundary conditions into account we choose its subspace with zero traces $W_0^{1,2}(I)$ for the space V. The space V is also reflexive separable Banach space, moreover it is a Hilbert space with scalar product $(u, v) = \int_I [u'v' + u\,v]\,\mathrm{d}x$.

Weak formulation.

The weak (often called generalized) formulation of the boundary value problem reads as follows:

Find $u \in V$ such that the identity (10.2) holds for each $v \in V$.

The weak formulation is in fact the abstract operator equation $A(u) = b$. Indeed, the operator A and the functional b are defined by relations

$$\begin{aligned} \langle A(u), v\rangle &= \int_I u'v'\,\mathrm{d}x + \int_I g(u)v\,\mathrm{d}x\,, \quad u, v \in V \\ \langle b, v\rangle &= \int_I fv\,\mathrm{d}x\,, \quad v \in V\,. \end{aligned}$$

Justification of the weak formulation.

We have to specify the assumptions for f in order to $b \in V'$ and to prove that the operator A maps V into V' , i. e. that for $u \in V$ the value $A(u) \in V'$.

We suppose that $f \in L^2(I)$. Then the functional b can be estimated using the Schwarz inequality (11.6) or (12.6):

$$|\langle b, v\rangle| = \left|\int_I fv\,\mathrm{d}x\right| \le \left[\int_I f^2\,\mathrm{d}x\right]^{1/2} \left[\int_I v^2\,\mathrm{d}x\right]^{1/2} =$$

$$= \|f\|_2 \cdot \|v\|_2 \le \mathrm{const} \cdot \|v\|_V\,.$$

The linear functional b is bounded, thus it is continuous, i. e. $b \in V'$.

Let us deal with the operator A. The form $\langle A(u), v\rangle$ is linear in v. It remains to verify that the functional $A(u)$ is continuous, i. e. $A(u) \in V'$. The operator A consists of two parts $A = A_1 + A_0$. The first part A_1 defined by

$$\langle A_1(u), v\rangle = \int_I u'\, v'\, \mathrm{d}x$$

maps V into V' by virtue of the estimate (12.6) yielding

$$|\langle A_1(u), v\rangle| = \left|\int_I u'v'\, \mathrm{d}x\right| \le \|u'\|_2 \cdot \|v'\|_2 \le \|u\|_V \cdot \|v\|_V\, .$$

Moreover, one can see that A_1 is a linear bounded operator and thus it is continuous. Since $[\langle A_1(u), u\rangle]^{1/2} = [\int_I {u'}^2\, \mathrm{d}x]^{1/2}$ is an equivalent norm on the space $V = W_0^{1,2}(I)$, (12.12), the linear operator A_1 is strongly monotone.

The second part A_0 of the operator A is defined by

$$\langle A_0(u), v\rangle = \int_I g(u)\, v\, \mathrm{d}x\, .$$

We need to prove that $A_0(u) \in V'$ for $u \in V$, i. e.

$$|\langle A_0(u), v\rangle| \le \mathrm{const} \cdot \|v\|_V\, , \quad \forall u, v \in V\, .$$

In the linear case (a) the estimate is clear. In the other cases we shall use the imbedding of Sobolev spaces, see (12.14) or e. g. [9].

$$V = W_0^{1,2}(I) \subset C^0(I)\, .$$

Indeed, the functions of V are absolutely continuous with $u(0) = 0$ and thus $u(x) = \int_0^x u'(s)\mathrm{d}s$ which implies using (11.8):

$$|u(x)| \le \int_0^x |u'(s)|\mathrm{d}s \le \int_I |u'|\mathrm{d}s \le \left[\int_I {u'}^2 \mathrm{d}s\right]^{1/2} \cdot [\mathrm{meas}(I)]^{1/2}$$

for all $x \in I$. Consequently

$$\max_I |u| \le \|u\|_V\, . \tag{10.4}$$

Since in all cases the function g is continuous, $g(u(\cdot))$ is also bounded. Thus we obtain the desired estimate

$$|\langle A_0(u), v\rangle| = \left|\int_I g(u) v\, \mathrm{d}x\right| \le \max_I |g(u)|\, \max_I |v| \le c(u)\, \|v\|_V$$

and the weak formulation is justified. Moreover, we have proved that in all cases the operator A_0 is bounded. Indeed, due to continuity of u, g and (10.4) the constant $c(u)$ depends on the norm $\|u\|$ and thus A_0 is bounded. Similarly, by the same argument, $u_n \to u$ in V implies $\|A_0 u_n - A_0 u\| \to 0$, i. e. A_0 is continuous.

Thus the weak formulation is justified and the opertor A is bounded and continuous.

Let us remark that the mapping $u \mapsto g(u)$ represents a special case of the so-called Nemytskij operator, see Section 13.

Application of abstract existence theorems.

Now we shall investigate the individual cases using abstract existence theorems. We shall use the fact that the seminorm $|u|_{1,2} = [\langle A_1(u), u\rangle]^{1/2}$ represents an equivalent norm on $W_0^{1,2}(I)$, see (12.13).

(a) The operator A is linear and continuous, thus it is Lipschitz continuous. For a non-negative constant c the operator A_0 is monotone and thus A is strongly monotone. Theorem on strongly monotone operator yields the existence of unique solution. Moreover, we have strong convergence of the approximate solutions given by (3.3) or strong convergence of Galerkin approximations due to Theorem 8.2, since the strongly monotone operator satisfies (S_0)-condition, see Theorem 15.1. Let us remark that if the constant c is negative, the solution need not exist.

(b) Again the operator A is continuous, bounded and strongly monotone, since A_0 is monotone

$$\langle A_0(u) - A_0(v), u - v\rangle = \int_I (u-v)^2(u^2 + uv + v^2)\,\mathrm{d}x \geq 0$$

Theorem 5.1 yields the existence of the solution. Moreover the solution is unique (Theorem 2.2) and the Galerkin approximations u_n converge strongly to u (Theorem 8.1, 8.2) since strongly monotone operator is strictly monotone and satisfies (S_0)-condition, see Lemma 15.1.

(c) Since g is a non-decreasing function, the operator A_0 is again monotone and we have the same result as in the case (b).

(d) Since g may have a decreasing segment, A_0 need not be monotone. Therefore, we make use of the fact that A_0 is strongly continuous. Indeed, let $u_n \overset{w}{\rightharpoonup} u$ in V. The compact imbedding $W_0^{1,2}(I) \subset\subset C^0(I)$, see (12.16) yields $u_n \to u$ in $C^0(I)$ strongly. Since g is continuous, we have $g(u_n) \to g(u)$ in $C^0(I)$. Thus $A_0(u_n) \to A_0(u)$ in V' which follows from the estimate

$$\|A_0(u_n) - A_0(u)\|'_V = \sup_{\|v\|\leq 1} \int_I [g(u_n) - g(u)]v\,\mathrm{d}x \leq$$

$$\leq \max_x |g(u_n(x)) - g(u(x))| \cdot \sup_{\|v\|\leq 1} \int_I |v|\,\mathrm{d}x \to 0$$

since $\int_I |v|\,\mathrm{d}x \le \|v\|$, see (11.8). Thus both operators A_0 and A_1 are weakly continuous.

To obtain the coercivity of A we have to add another assumption for g:

$$\liminf_{|\xi|\to\infty} g(\xi)\operatorname{sign}\xi > -\infty \tag{10.5}$$

The condition ensures that A_0 is not "too negative", i. e. it does not violate the coercivity of A.

Due to Theorem 6.1 we arrived to the conclusion:

If the assumption (10.5) is satisfied then the problem admits a solution. The solution need not be unique.

If (10.5) is not satisfied, the operator A need not be coercive and the problem need not have a solution for some right-hand sides f, see [5], Chapter VI.

Let us remark that the operator is potential and the problem can be studied by means of variational methods with similar results, see [5], Theorem 26.13.

Example 2. **General ordinary differential equation.**

We shall consider a general second order differential equation in divergent form with Dirichlet boundary conditions:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[a_1(x,u(x),u'(x))\right] + a_0(x,u(x),u'(x)) = f(x) \qquad \text{in } I = (0,1)$$

$$u(0) = u(1) = 0\,.$$

Weak formulation.

We rewrite the problem in the form of an operator equation on a Banach space. For a second order problem we shall use the Sobolev space $W^{1,2}(I)$ as the basic space. Due to stable boundary conditions $u(0) = u(1) = 0$ we shall take test functions from its subspace with zero traces $V = W_0^{1,2}(I)$. It is a reflexive separable Banach space, too.

The equation is in the divergent form, hence multiplying it by v and integrating the first term by parts (due to $v(0) = v(1) = 0$ the boundary terms vanish) we obtain the integral identity

$$\int_I [a_1(\cdot,u,u')\,v' + a_0(\cdot,u,u')\,v]\,\mathrm{d}x = \int_I f\,v\,\mathrm{d}x\,. \tag{10.6}$$

We define the operator $A : V \to V'$ by the relation

$$\langle A(u), v\rangle = \int_I [a_1(\cdot,u,u')v' + a_0(\cdot,u,u')\,v]\,\mathrm{d}x\,, \quad \forall\, u,v \in V\,. \tag{10.7}$$

We can consider a more general right-hand side $f = f_0 - f_1'$. Since we admit also discontinuous f_1 the case includes even the Dirac distribution in the right-hand side f. We define $b \in V'$ by the relation

$$\langle b, v \rangle = \int_I [f_0\, v + f_1\, v']\, \mathrm{d}x\,, \quad v \in V\,. \tag{10.8}$$

The weak formulation of the problem reads as follows:

$$\text{Find } u \in V \text{ such that } \quad \langle A(u), v \rangle = \langle b, v \rangle \quad \text{holds for all } v \in V\,. \tag{10.9}$$

Justification of the weak formulation.

We have to specify the coefficients in such a way that the integrals in the formulation exist and are finite, in other words that the operator A defined above really acts from V into V' and $b \in V'$.

We assume $f_0, f_1 \in L^2(I)$. Due to the estimate (12.6) we have

$$|\langle b, v \rangle| \le \|f_0\|_2\, \|v\|_2 + \|f_1\|_2\, \|v'\|_2 \le \text{const.} \|v\|_V$$

which yields the functional $b : V \to R$ is continuous, i. e. $b \in V'$.

Let us deal with the operator A. We have to find conditions which are general enough and that ensure the composed functions $a_i(\cdot, u(\cdot), u'(\cdot))$ are measurable and integrable such that A acts from V into V'. Let us remark that superposition of measurable functions need not be measurable.

The problem can be solved by Theorem on Nemytskij operators, see Section 13.

First, we adopt a natural assumption that the coefficients a_0, a_1 satisfy the so-called Carathéodory conditions:

$$a_i(x, \xi_0, \xi_1) \left\{ \begin{array}{l} \text{are measurable in } x \text{ for all } \xi \in \mathbb{R}^2 \text{ and} \\ \text{continuous in } \xi \text{ for almost all } x \in I\,. \end{array} \right. \tag{10.10}$$

In the integral identity we have $\int_I a_0(\cdot, u, u')\, v\, \mathrm{d}x$ and $\int_I a_1(\cdot, u, u')\, v'\, \mathrm{d}x$. Since $v, v' \in L^2(I)$, we need $a_i(\cdot, u, u') \in L^2(I)$, i. e. we need the mappings

$$(u, u') \in L^2(I) \times L^2(I) \quad \mapsto \quad a_i(\cdot, u, u') \in L^2(I)\,.$$

Following the theorem it is ensured by the growth condition

$$|a_i(x, \xi_0, \xi_1)| \le g(x) + c(|\xi_0| + |\xi_1|)\,, \quad i = 0, 1\,, \tag{10.11}$$

where $g \in L^2(I)$, $c > 0$. Indeed, the estimate yields

$$|a_i(x, u(x), u'(x))|\, \mathrm{d}x \le g(x) + c(|u(x)| + |u'(x)|)\,,$$

which implies using (11.4) the estimate

$$\int |a_i(x, u(x), u'(x))|^2 \, \mathrm{d}x \le 3 \left[\|g\|_2^2 + c^2 \|u\|_2^2 + c^2 \|u'\|_2^2 \right] .$$

Condition (10.11) can be weakened if we take into account that the function from V has its values in a better space than L^2 and use the imbedding of Sobolev spaces. In our case $V \subset W^{1,2}(I) \subset C^0(I) \subset L^\infty(I)$, see (12.14). Thus $v \in L^\infty(I)$ and it is sufficient to require only $a_0(\cdot, u, u') \in L^1(I)$. The necessary mappings

$$\begin{aligned} (u, u') \in L^\infty \times L^2(I) \quad &\mapsto \quad a_1(\cdot, u, u') \in L^2(I) , \\ (u, u') \in L^\infty \times L^2(I) \quad &\mapsto \quad a_0(\cdot, u, u') \in L^1(I) . \end{aligned}$$

are ensured by the growth conditions

$$\begin{aligned} |a_1(x, \xi_0, \xi_1)| \quad &\le c_1(|\xi_0|)(g_1(x) + |\xi_1|) , \\ |a_0(x, \xi_0, \xi_1)| \quad &\le c_0(|\xi_0|)(g_0(x) + |\xi_1|^2) , \end{aligned} \tag{10.12}$$

where $c_0(t), c_1(t)$ are continuous functions and $g_0 \in L^1(I)$, $g_1 \in L^2(I)$.

We can conclude:

Let the coefficients $a_i(x, \xi_0, \xi_1)$ satisfy Carathéodory conditions and (10.11) or (10.12) and $f_0, f_2 \in L^2(I)$. Then the operator A maps V into V' and the weak formulation (10.9) of the problem is justified.

Application of the abstract theorems.

Due to the Theorem on Nemytskij operators — assertion (c), the operator A is continuous and bounded.

The coercivity of the operator can be ensured by the condition

$$a_1(x, \xi_0, \xi_1)\xi_1 + a_0(x, \xi_0, \xi_1)\xi_0 \ge c\,|\xi_1|^2 - K , \quad \forall \xi_0, \xi_1 \in \mathbb{R} \tag{10.13}$$

holding for a. e. $x \in I$, where $c > 0$, $K \in R$. Indeed, the condition yields

$$\langle A(u), u \rangle = \int_I [a_1(\cdot, u, u')u' + a_0(\cdot, u, u')u] \, \mathrm{d}x \ge c \int_I {u'}^2 \, \mathrm{d}x - K .$$

Since $[\int {u'}^2 \, \mathrm{d}x]^{1/2}$ is an equivalent norm on V, the operator A is coercive.

Concerning the other assumption we distinguish four cases:

(a) Monotone case. The monotony of the operator A can be ensured by the condition

$$[a_1(x,\xi_0,\xi_1) - a_1(x,\eta_0,\eta_1)]\,(\xi_1-\eta_1) + [a_0(x,\xi_0,\xi_1) - a_1(x,\eta_0,\eta_1)]\,(\xi_0-\eta_0) \geq 0$$

$$\forall\,\xi_0,\xi_1,\eta_0,\eta_1 \in \mathbb{R} \text{ and for a.e. } x \in I\,. \tag{10.14}$$

Indeed, it yields

$$\langle A(u)-A(v), u-v\rangle = \int_I \{[a_1(\cdot,u,u') - a_1(\cdot,v,v')]\,(u'-v') +$$

$$+ [a_0(\cdot,u,u') - a_0(\cdot,v,v')]\,(u-v)\}\,\mathrm{d}x \geq 0\,.$$

Using Theorem 5.1 we obtain the conclusion:

Let the assumptions (10.10) and (10.11) or (10.12) with (10.13), (10.14) be satisfied and $f_0, f_1 \in L^2(I)$. *Then the problem (10.9) has a solution. All solutions form a nonempty closed convex subset of* V.

(b) Weakly continuous case. In many important cases monotony is not satisfied. If the operator is quasilinear i. e. the coefficient a_1 is linear in ξ_1:

$$a_1(x,\xi_1,\xi_0) = \xi_1 \cdot a_1^*(x,\xi_0)\,, \tag{10.15}$$

and $a_0(x,\xi_1,\xi_0)$ is linear in ξ_1 or independent of ξ_1 then due to Lemma 17.3 both parts of the operator A are weakly continuous and Theorem 6.1 yields:

Let the assumptions (10.10), (10.12), (10.13) and (10.15) be satisfied and $f_0, f_1 \in L^2(I)$. *Then the problem (10.9) admits a solution.*

(c) The (S_+) condition case. In many important cases the operator A is neither monotone nor quasilinear. Instead of it we can assume strong monotony in the principal part of the operator, i. e.

$$[a_1(x,\theta_0,\xi_1) - a_1(x,\theta_0,\eta_1)](\xi_1-\eta_1) \geq \alpha|\xi_1-\eta_1|^2 \quad (\alpha>0)\,. \tag{10.16}$$

Then the operator A_1, given by $\langle A_1(u), v\rangle = \int_I a_1(\cdot,u,u')v'\,\mathrm{d}x$, satisfies condition $(S)_+$. Indeed, let $u_n \overset{w}{\rightharpoonup} u$ in V and let $\limsup\langle A_1(u_n) - A_1(u), u_n - u\rangle \leq 0$. Due to (10.16) we obtain

$$\alpha\|u_n' - u'\|_{L^2}^2 \leq \int_I [a_1(\cdot,u_n,u_n') - a_1(\cdot,u_n,u')](u_n'-u')\,\mathrm{d}x =$$

$$= \langle A_1(u_n) - A_1(u), u_n-u\rangle + \int_I [a_1(\cdot,u,u')](u_n'-u')\,\mathrm{d}x\,.$$

Let us pass to the $\limsup$. Due to the assumption, $\limsup$ of the first term is non-positive. Since $u_n \overset{w}{\rightharpoonup} u$ in V, the compact imbedding of V into $C^0(I)$ implies uniform convergence $u_n \to u$ and due to the continuity of $a_1(x,\xi_0,\xi_1)$ in ξ_0 the second integral tends to zero. Since $\|u'\|_2 = |u|_{1,2}$ is an equivalent norm on V, we obtain $\|u_n - u\|_V^2 \to 0$ which proves condition $(S)_+$.

Further, we suppose that a_0 is independent of ξ_1, i. e.

$$a_0 = a_0(x, \xi_0)\,. \tag{10.17}$$

Then the second part of A, the operator A_0 is strongly continuous. Due to Lemma 16.2 (c) the sum A also satisfies (S_+). Justifying the formulation we ensured that the operator is continuous and bounded. Using Lemma 16.1 we see that it satisfies (M_0)-condition. Thus the assumptions of Theorem 7.1 are satisfied and with Theorem 8.2 we reach the following conclusion:

Let the assumptions (10.10), (10.12), (10.13), (10.16) and (10.17) be satisfied and $f_0, f_1 \in L^2(I)$. Then the problem (10.9) has the solution. Moreover the sequence of Galerkin approximate solutions contains a strongly converging subsequence.

(d) Pseudomonotone case. Let us briefly mention the most general case. We assume only strict monotony in the principal part of the operator A, i. e.

$$[a_1(x, \theta_0, \xi_1) - a_1(x, \theta_0, \eta_1)](\xi_1 - \eta_1) > 0 \quad \forall\, \theta, \xi_1, \eta_1, \quad \xi_1 \neq \eta_1\,. \tag{10.18}$$

Then one can prove the pseudomonotony of the operator A without assumption (10.17), see [10], [13].

Remarks.

(a) If the coefficients $a_i(x, \xi_0, \xi_1)$ are differentiable in ξ_0, ξ_1, then the monotony condition (10.14) can be rewritten in the form

$$\frac{\partial a_1}{\partial \xi_1}(x, \xi_0, \xi_1)\eta_1^2 + \left[\frac{\partial a_1}{\partial \xi_0}(x, \xi_0, \xi_1) + \frac{\partial a_0}{\partial \xi_1}(x, \xi_0, \xi_1)\right]\eta_0\eta_1 +$$

$$+ \frac{\partial a_0}{\partial \xi_0}(x, \xi_0, \xi_1)\eta_0^2 \geq 0 \quad \forall\, \xi_0, \xi_1, \eta_0, \eta_1 \in \mathbb{R}\,.$$

Indeed, using the mean value theorem we can write

$$[a_1(x, \xi_0, \xi_1) - a_1(x, \eta_0, \eta_1)](\xi_1 - \eta_1) + [a_0(x, \xi_0, \xi_1) - a_0(x, \eta_0, \eta_1)](\xi_0 - \eta_0) =$$

$$= \int_0^1 \left[\frac{\partial a_1}{\partial \xi_1}(\theta)(\xi_1 - \eta_1) + \frac{\partial a_1}{\partial \xi_0}(\theta)(\xi_0 - \eta_0)\right](\xi_1 - \eta_1)\mathrm{d}t +$$

$$+ \int_0^1 \left[\frac{\partial a_0}{\partial \xi_1}(\theta)(\xi_1 - \eta_1) + \frac{\partial a_0}{\partial \xi_0}(\theta)(\xi_0 - \eta_0)\right](\xi_0 - \eta_0)\mathrm{d}t\,,$$

where θ stands for $(x, \eta_0 + t(\xi_0 - \eta_0), \eta_1 + t(\xi_1 - \eta_1))$. Thus the above introduced condition yields monotony (10.14). Similarly we can rewrite assumptions (10.13), (10.16) or (10.18).

(b) The fact that only homogeneous Dirichlet boundary condition were considered is not substantial. Other boundary conditions bring only technical difficulties.

(c) The problems with coefficients growing more rapidly than (10.12) can be investigated using Sobolev spaces $W^{1,p}(I)$ with $p > 2$ or using Orlicz spaces, see e.g. [5], [9].

(d) The above introduced procedure can be applied also to boundary value problems for differential equations of order $2m$, for partial differential equations and even for systems of equations, see [5], [10], [12] .

Example 3. **Nonlinear heat-conduction equation.**

Stationary heat-conduction equation is a well known linear elliptic second order equation. If the constant describing heat conductivity properties depends on the temperature, we obtain a nonlinear problem which is not monotone. Since it is quasilinear we can apply theorem on weakly continuous operators.

Formulation of the problem.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N = 2$ or $3)$ with Lipschitz boundary $\partial\Omega$ divided into two parts Γ_0, Γ_1. We assume that Γ_0 has positive surface measure.

We shall consider the equation

$$-\sum_i \frac{\partial}{\partial x_i}\left[a(x,u)\frac{\partial u}{\partial x_i}\right] = f \quad \text{in } \Omega \tag{10.19}$$

with mixed boundary condition

$$u = 0 \quad \text{on } \Gamma_0\,, \qquad \sum_i a(x,u)\frac{\partial u}{\partial x_i} n_i = g \quad \text{on } \Gamma_1\,. \tag{10.20}$$

Let us remind that the problem describes steady state of the heat conduction ($u(x)$ means temperature) in a body occupying volume Ω with internal heat sources f. Function $a(x,\xi)$ describes heat conduction properties of the material. On the boundary, temperature or heat flow is prescribed. To simplify the problem we consider zero stable boundary conditions only. The nonhomogeneous case $u = U_0$ on Γ_0 causes technical difficulties only.

Integral identity.

Due to prescribed value at Γ_0 we multiply the equation by a function v satisfying $v = 0$ on Γ_0. Using Green's theorem (see Part III) we transform the left-hand side integral. In the integral over Γ_1 we use the boundary condition. In this way we obtain the integral identity

$$\int_\Omega \sum_i a(x,u) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_1} g\, v \, \mathrm{d}S \quad v \in V\,.$$

Weak formulation.

The space V is defined as the closure of the set $\{u \in C^1(\overline{\Omega}),\, u = 0 \quad \text{on } \Gamma_0\}$ in the norm of Sobolev space $W^{1,2}(\Omega)$. The space V is a reflexive separable Banach space.

We define the operator $A\, V \to V'$ and the functional $b \in V'$ by the relations

$$\langle A(u), v\rangle = \int_\Omega \sum_i a(x,u) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x\,, \quad \forall\, u, v \in V\,, \tag{10.21}$$

$$\langle b, v\rangle = \int_\Omega f v \, \mathrm{d}x + \int_{\Gamma_1} g\, v \, \mathrm{d}S\,, \quad \forall\,, v \in V\,. \tag{10.22}$$

The problem can be formulated as follows:

$$\text{Find } u \in V \text{ such that} \quad \langle A(u), v\rangle = \langle b, v\rangle \quad \forall v \in V\,. \tag{10.23}$$

Justification and application of abstract existence theorem.

To ensure $b \in V'$ we assume

$$f \in L_2(\Omega)\,, \qquad g \in L_2(\Gamma_1)\,. \tag{10.24}$$

Further, we use the Lemma 17.3 to prove that the operator A being of the form (17.1) is "well" defined and weakly continuous:

We assume that the coefficient $a(x,\xi)$ satisfies the Carathéodory conditions. Since the derivatives of u and v are in $L^2(\Omega)$ we need $a(\cdot,u)$ to be in $L^\infty(\Omega)$. This is ensured by the growth condition (see Section 13, assertion (b")

$$|a(x,\xi)| \le c \qquad (c < \infty)\,. \tag{10.25}$$

Since the imbedding $W^{1,2}(\Omega) \subset\subset L^2(\Omega)$ is compact for $N = 2, 3$ each term is well defined and weakly continuous. The same holds for their sum, Lemma 17.1.

It remains to verify the coercivity of operator A. It is ensured by the assumption

$$a(x,\xi) \ge \alpha \qquad (\alpha > 0) \tag{10.26}$$

since $|\cdot|_{1,2}$ represents an equivalent norm on V.

We can conclude: *If (10.24)–(10.26) is satisfied then the problem (10.23) has a solution.*

Remark. For (10.25) the operator $u \mapsto a(\cdot, u)$ mapping $L^p(\Omega)$ into $L^\infty(\Omega)$ is bounded but not continuous. Considering the operator with the function $a(x, \xi_0) \cdot \xi_1)$ we obtain desired continuity. Indeed, mapping $u, \frac{\partial u}{\partial x_i} \mapsto a(\cdot, u)\frac{\partial u}{\partial x_i}$ is continuous, since the growth condition (13.4)

$$a(x, \xi_0) \cdot \xi_1 \le c \cdot |\xi_1|$$

is satisfied.

Example 4. **Stationary Navier-Stokes equations.**

The system of Navier-Stokes equations is a nonlinear problem due to nonlinear convective term. It represents a simple model of stationary viscous flow. The problem leads to an abstract equation with weakly continuous operator.

Formulation of the problem.

Let Ω be a bounded domain in R^N ($N = 2$ means the plane case, $N = 3$ the space case) with Lipschitz boundary Γ. We shall consider the following system of equations called stationary Navier-Stokes equations:

$$(10.27) \qquad -\nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = f_i + \frac{\partial p}{\partial x_i} \qquad i = 1, \dots, N \quad \text{in } \Omega\,,$$

$$(10.28) \qquad \sum_i \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega\,,$$

where the sums are from 1 to N. For the sake of simplicity we consider homogeneous Dirichlet boundary condition

$$(10.29) \qquad u = 0 \quad \text{on } \Gamma\,.$$

The system describes the steady state flow of the incompressible viscous liquid occupying volume Ω subjected to given external volume forces $f \equiv \{f_i(x)\}$. The introduced boundary condition (10.28) means that the liquid is closed within fixed walls.

The flow is described by two unknowns: the vector function velocity $u \equiv \{u_i(x)\}_i$ and the scalar function pressure $p \equiv p(x)$.

The first equation is the equation of motion: the first term on the left-hand side is the viscous term with ν — the constant of viscosity, the second one is the convection term. The second equation is the continuity equation expressing the mass conservation law.

Weak formulation.

We shall look for the solution in the space V defined as the closure of the set

$$\left\{ u \in \left[C^1(\Omega)\right]^N , \quad \sum_i \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \Gamma \right\}$$

in the norm of vector Sobolev space $[W^{1,2}(\Omega)]^N$. The space V is a separable reflexive Banach space.

We multiply the i-th equation (10.27) by a function v_i, integrate it over Ω and sum them up. Applying the Green theorem to the viscous term and term with the pressure we obtain

$$-\nu \int_\Gamma \sum_{i,j} \frac{\partial u_i}{\partial x_j} v_i n_j \,\mathrm{d}S + \nu \int_\Omega \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}x + \int_\Omega \sum_{i,j} u_j \frac{\partial u_i}{\partial x_j} v_i \,\mathrm{d}x =$$

$$= \int_\Omega \sum_i f_i v_i \,\mathrm{d}x + \int_\Gamma p \sum_i v_i n_i \,\mathrm{d}S - \int_\Omega p \sum_i \frac{\partial v_i}{\partial x_i} \,\mathrm{d}x\,.$$

Taking $u, v \in V$ the integrals over Γ vanish due to $v = 0$ on Γ. The integral with pressure p vanishes due to $\sum(\partial v_i)/(\partial x_i) = 0$. Moreover, $u \in V$ implies that the equation (10.28) is satisfied; thus it can be omitted.

Using a bilinear form $a(\cdot,\cdot)$ and a trilinear form $b(\cdot,\cdot,\cdot)$

$$a(u,v) = \int_\Omega \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}x\,, \qquad b(u,v,w) = \int_\Omega \sum_{i,j} u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x\,,$$

we define the operator $A : V \to V'$ and the functional $f \in V'$

$$(10.30) \qquad \langle A(u), v\rangle = a(u,v) + b(u,u,v)\,, \qquad \langle f, v\rangle = \int_\Omega \sum_i f_i v_i \,\mathrm{d}x\,.$$

In the introduced notation the problem can be formulated as follows:

$$(10.31) \qquad \text{Find } u \in V \text{ such that} \quad \langle A(u), v\rangle = \langle f, v\rangle \quad \forall v \in V\,.$$

Remark. The weak formulation does not contain the pressure p. One can prove that to any sufficiently smooth weak solution u of (10.31) there exists a function p such that u, p is the solution to (10.27)–(10.29).

Justification and application of abstract existence theorem.

In order to ensure $f \in V'$ we assume $f \in [L_2(\Omega)]^N$.

The bilinear form $a(\cdot,\cdot)$ is continuous on $V \times V$. Following Lemma 17.2 the first part of A is weakly continuous.

The trilinear form $b(\cdot,\cdot,\cdot)$ consists of terms $\int u_j\,(\partial u_i)/(\partial x_j)\,v_i\,\mathrm{d}x$ which are of type (17.2). Using Lemma 17.3 we prove that they are well defined and weakly continuous. Indeed, for $N \le 3$ the imbedding

$$W^{1,2}(\Omega) \subset\subset L^4(\Omega)$$

is compact, see (12.15). Thus the problem (10.31) is well defined.

It remains to prove that the operator A is coercive. We get use of the equality

$$b(u,u,u) = 0 \qquad u \in V\,. \tag{10.32}$$

Applying the Green theorem for $v = 0$ on Γ we obtain

$$b(u,v,w) = \int_\Omega \sum_{i,j} u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x = -\int_\Omega \sum_{i,j} \frac{\partial u_j}{\partial x_j} v_i w_i \,\mathrm{d}x - \int_\Omega \sum_{i,j} u_j v_i \frac{\partial w_i}{\partial x_j}\,\mathrm{d}x\,.$$

The first integral on the right-hand side vanishes due to $u \in V$. Thus we obtained $b(u,v,w) = -b(u,w,v)$ and for $v = w$ the equality (10.32) follows.

Taking (10.32) into account we have $\langle A(u), u\rangle = a(u,u) \ge \mathrm{const}\cdot\|u\|_V^2$ since $[a(u,u)]^{1/2}$ forms an equivalent norm on V and the coerciveness follows.

We can conclude: *For $f \in [L_2(\Omega)]^N$ the problem (10.31) admits a solution.*

Remark. Nonhomogeneous boundary conditions cause some difficulties in the proof of coerciveness of the operator, a special "cut off" function should be used. For small f also uniqueness can be proved.

Example 5. **General partial differential equation.**

The following example is a general second order partial differential equation. The application of the method will be only outlined.

Formulation of the problem.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$ divided into two parts Γ_0, Γ_1 and let us consider the equation

$$-\sum_i \frac{\partial}{\partial x_i}\,[a_i(x,u,\nabla u)] + a_0(x,u,\nabla u) = f \quad \text{in } \Omega \tag{10.33}$$

with mixed boundary conditions

$$u = u_0 \quad \text{on } \Gamma_0\,, \tag{10.34}$$

$$\sum_i a_i(x,u,\nabla u)\,n_i = g \quad \text{on } \Gamma_1\,. \tag{10.35}$$

Weak formulation and its justification.

Taking into account the stable boundary condition (10.34) we define the Banach space V as the closure of the set $\{u \in C^1(\overline{\Omega}), u = 0 \quad \text{on } \Gamma_0\}$ in the Sobolev space $W^{1,2}(\Omega)$. We define the operator $A : W^{1,2}(\Omega) \to V'$ by

$$(10.36) \qquad \langle A(u), v\rangle = \int_\Omega \left[\sum_i a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(x, u, \nabla u)\, v \right] \mathrm{d}x$$

and the functional $b \in V'$ by

$$(10.37) \qquad \langle b, v\rangle = \int_\Omega f v \,\mathrm{d}x + \int_{\Gamma_1} g v \,\mathrm{d}S\,.$$

Thus we obtain weak formulation of the problem (10.33)-(10.35):

$$(10.38) \qquad \begin{array}{l} \text{Find } u \in W^{1,2}(\Omega) \text{ such that } u - u_0 \in V \text{ and} \\ \langle A(u), v\rangle = \langle b, v\rangle \quad \text{holds for each } v \in V\,. \end{array}$$

To justify this formulation we adopt assumptions

$$(10.39) \qquad u_0 \in W^{1,2}(\Omega)\,, \quad f \in L^2(\Omega)\,, \quad g \in L^2(\Gamma_1)\,.$$

According to Theorem on Nemytskij operators it is sufficient to suppose that the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\,, i = 0, 1, \ldots, N$ satisfy the Carathéodory conditions (13.1) and the growth conditions

$$(10.40) \qquad |a_i(x, \xi_0, \xi_1, \ldots, \xi_N)| \le g_i(x) + c_i \sum_{j=0}^N |\xi_j|\,,$$

where $g_i \in L^2(\Omega)$ and $c_i > 0$. This growth restriction can be weakened using imbeddings of Sobolev spaces, see (12.13).

Application of the monotone operators.

The above introduced assumptions yield also boundedness and continuity of the operator A. The condition

$$(10.41) \qquad \sum_{i=0}^N a_i(x, \xi_0, \xi_1, \ldots, \xi_N)\xi_i \ge d_1 \sum_{i=1}^N \xi_i^2 + d_0\xi_0^2 - h(x)$$

with $d_0, d_1 > 0 \;\; h \in L^1(\Omega)$ implies the coercivity of the operator. If $|u|_{1,2}$ is an equivalent norm on V, then we can admit $d_0 = 0$.

Further, if the following condition

$$\sum_{i=0}^{N}(a_i(x,\xi_0,\xi_1,\dots,\xi_N)-a_i(x,\eta_0,\eta_1,\dots,\eta_N))(\xi_i-\eta_i)\ge 0\,, \tag{10.42}$$

is satisfied, then the operator is monotone and using Theorem 5.1 we obtain the following result:

Let (13.1), (10.39)–(10.42) be satisfied. Then the problem (10.38) has a solution. The solutions form a closed convex subset.

If (10.42) is not satisfied, we can assume only strict monotony in the principal part of the operator, i. e.

$$\sum_{i=1}^{N}(a_i(x,\theta_0,\xi_1,\dots,\xi_N)-a_i(x,\theta_0,\eta_1,\dots,\eta_N))(\xi_i-\eta_i)>0\,,$$

$$\text{for all}\quad \theta_0\,,\quad (\xi_1,\dots,\xi_N)\ne(\eta_1,\dots,\eta_N)\,. \tag{10.43}$$

Then we can prove the pseudomonotony which implies (M_0)-condition. For the proof we refer to [13]. Thus using Theorem 7.1 we obtain the following result:

Let (13.1), (10.39)–(10.41) and (10.43) be satisfied. Then the problem (10.38) admits a solution.

Remarks.

(a) If the operator is quasilinear, i. e.

$$\langle A(u),v\rangle=\int_\Omega\left[\sum_{i,j}\frac{\partial u}{\partial x_j}a_{i,j}(x,u)\frac{\partial v}{\partial x_i}+a_0(x,u,\nabla u)\,v\right]\mathrm{d}x$$

with convenient growth condition, then it is weakly continuous. If it is in addition coercive, e. g. (10.41) holds, then by virtue of Theorem 6.1 the solution exists.

(b) The same procedure can be applied to a system of equations, see e. g. [10]; the obtained results are similar, only the formulae have more indices.

(c) If the coefficients are differentiable, the conditions (10.41)–(10.43) are often expressed in terms of derivatives.

EXAMPLE 6. **The $2m$-order partial differential equation.**

To close the examples we shall briefly mention the case of $2m^{th}$order equation

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} \mathrm{D}^\alpha [a_\alpha(x,u,\mathrm{D}^1 u,\dots,\mathrm{D}^m u)] = f \quad \text{on } \Omega \tag{10.44}$$

with suitable boundary conditions, see e. g. [5]. The simplest case is

$$D^\beta u = 0 \quad \text{on } \partial\Omega\,, \quad \forall\,\beta\,, \quad |\beta| \le m-1\,. \tag{10.45}$$

In this example we use the notation with multiindices denoted by Greek letters $\alpha = (\alpha_1,\dots,\alpha_N)$, $\alpha_i \in \{0,1,\dots,m\}$. We put $|\alpha| = \sum_i \alpha_i$, and further D^α means
$\partial^{|\alpha|}/(\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N})$ and $\mathrm{D}^k u$ denotes $\{\mathrm{D}^\alpha u\,, |\alpha| = k\}$.

The suitable Banach space V is a subspace of the Sobolev space $W^{m,2}(\Omega)$ containing $W_0^{m,2}(\Omega)$ chosen in accordance with the consedered boundary conditions. In case (10.45) we choose $V = W_0^{m,2}(\Omega)$. The corresponding operator $A : W^{m,2}(\Omega) \to V'$ is defined by

$$\langle A(u), v\rangle = \int_\Omega \sum_{|\alpha|\le m} a_\alpha(x,u,\mathrm{D}^1 u,\dots,\mathrm{D}^m u)\mathrm{D}^\alpha v)\,\mathrm{d}x\,. \tag{10.46}$$

In order to the operator A be "well" defined, the coefficients α_k are supposed to satisfy the Carathéodory conditions and the growth conditions

$$|a_\alpha(x,\xi)| \le g(x) + c \sum_{|\beta|\le m} |\xi_\beta| \quad \forall \xi = (\xi_\beta,\, |\beta| \le m)\,. \tag{10.47}$$

Again using the imbedding theorems, this growth conditions can be weakened, see e. g. [5], Theorem 16.14. The condition

$$\sum_{|\alpha|\le m} a_\alpha(x,\xi)\xi_\alpha \ge d_1 \sum_{|\alpha|=m} \xi_\alpha^2 \,+\, d_0\,\xi_0^2 \;=\; h(x) \tag{10.48}$$

implies coercivity, the condition

$$\sum_{|\alpha|\le m} [a_\alpha(x,\xi) - a_\alpha(x,\eta)](\xi_\alpha - \eta_\alpha) \ge 0 \tag{10.49}$$

ensures monotony of the operator and the existence result follows. It is sufficient to assume only strict monotone in the principal part, i. e.

$$\sum_{|\alpha|=m} [a_\alpha(x,\hat\theta,\xi_m) - a_\alpha(x,\hat\theta,\eta_m)](\xi_\alpha - \eta_\alpha) \ge 0 \quad \forall\,\xi_m \ne \eta_m\,, \tag{10.50}$$

where $\hat\theta = (\theta_\beta,\, |\beta| \le m-1)$, $\xi_m = (\xi_\beta, |\beta| = 0)$, which implies pseudomonotony. However, the proof is rather complicated, see [13]. In this way the abstract existence theorems can be applied.

Part III

Auxiliary results

In this last part we introduce several auxiliary results that may be useful in application of the abstract results to particular problems. The results are often without proof.

We start with Green's formula. Integration by parts in dimension one

$$\int_a^b f' g \,\mathrm{d}x = \big[f\, g \big]_{x=a}^b - \int_a^b f\, g' \,\mathrm{d}x$$

can be generalized to higher dimension:

Green's theorem. *Let Ω be a domain with Lipschitz boundary and u, v smooth function on $\overline{\Omega}$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x = \int_{\partial\Omega} u\, v\, n_i \,\mathrm{d}S - \int_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x\,,$$

where n_i is i-th component of the outer unit normal vector.

11. Inequalities.

Justifying the weak formulation we often need to estimate integrals either in cases of linear operators or in growth condition for Nemytskij operators.

Inequalities for finite sequences.

The simplest inequality

$$|a\, b| \le \frac{1}{2}\left(a^2 + b^2\right) \tag{11.1}$$

is a consequence of the inequality $(a-b)^2 \ge 0$. Similar inequality $(\varepsilon a - b/\varepsilon)^2 \ge 0$ yields

$$|a\, b| \le \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2\,. \tag{11.2}$$

Also the inequality

$$(a+b)^2 \le 2(a^2 + b^2) \tag{11.3}$$

is another consequence of (11.1). It can be generalized to n members

$$(a_1 + a_2 + \cdots + a_n)^2 \le n(a_1^2 + a_2^2 + \cdots + a_n^2)\,. \tag{11.4}$$

Again, the proof follows from (11.1) applied to terms $2a_i a_j$. Similar estimate holds also for p^{th}-power ($p > 0$):

$$|a_1 + a_2 + \cdots + a_n|^p \le c \cdot (|a_1|^p + |a_2|^p + \cdots + |a_n|^p)\,, \tag{11.5}$$

where c is a positive constant dependent on p, n. Let us introduce a simple proof with $c = n^p$. Let $M = \max(|a_1|, \ldots, |a_n|)$. Since $|a_i/M| \le 1$ we have

$$|a_1 + \cdots + a_n|^p = \left|\frac{a_1}{M} + \cdots + \frac{a_n}{M}\right|^p (M)^p \le n^p\,(M)^p \le n^p\,(|a_1|^p + \cdots + |a_n|^p)\ .$$

The estimate holds with a better constant c but the proof is more complicated.

Inequalities for integrals.

We shall suppose that all expressions make sense, particularly $\Omega \subset \mathbb{R}^N$ and the integrals are finite.

The most frequent is the Schwarz inequality:

$$\left|\int_\Omega f\,g\,\mathrm{d}x\right| \le \left[\int_\Omega |f|^2\,\mathrm{d}x\right]^{1/2} \cdot \left[\int_\Omega |g|^2\,\mathrm{d}x\right]^{1/2} \tag{11.6}$$

which is a special case of the Hölder inequality

$$\left|\int_\Omega f\,g\,\mathrm{d}x\right| \le \left[\int_\Omega |f|^p\,\mathrm{d}x\right]^{1/p} \cdot \left[\int_\Omega |g|^{p'}\,\mathrm{d}x\right]^{1/p'}\,, \tag{11.7}$$

where p, p' are conjugate exponents $p, p' \in (1, \infty)$ satisfying $1/p + 1/p' = 1$, i. e. $p' = p/(p-1)$. For proof see e. g. [9]. Putting $g = 1$ we obtain:

$$\left|\int_\Omega f\,\mathrm{d}x\right| \le \left[\int_\Omega |f|^p\,\mathrm{d}x\right]^{1/p} [\mathrm{meas}(\Omega)]^{(p-1)/p}\ . \tag{11.8}$$

The Hölder inequality can be generalized to a product of finite number of functions $f_1, f_2, \ldots, f_k$:

$$\left|\int_\Omega f_1 \cdots f_k\,\mathrm{d}x\right| \le \left[\int_\Omega |f_1|^{p_1}\,\mathrm{d}x\right]^{1/p_1} \cdots \left[\int_\Omega |f_k|^{p_k}\,\mathrm{d}x\right]^{1/p_k}\,, \tag{11.9}$$

where the exponents $p_i \in (1, \infty)$ satisfy

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1\,.$$

The inequality can be proved by multiple use of the Hölder inequality.

12. Function spaces.

We shall briefly introduce basic properties of Lebesgue and Sobolev spaces. Let Ω be a domain in $\mathbb{R}^N$ (although the Lebesgue spaces can be introduced for any measurable set) and let $\mathcal{M}(\Omega)$ denote the set of all measurable functions on Ω.

Lebesgue spaces.

Lebesgue spaces of integrable functions on Ω have one parameter $p \in [1, \infty]$. For $p < \infty$ the L^p norm of a measurable function u is defined by

$$\|u\|_p = \left[\int_\Omega |u|^p \, \mathrm{d}x \right]^{1/p} , \tag{12.1}$$

and for $p = \infty$ it is

$$\|u\|_\infty = \operatorname{ess\,sup}\{ |u(x)| \mid x \in \Omega \} \equiv \inf_{|N|=0} \sup_{x \in \Omega - N} |u(x)| \, . \tag{12.2}$$

Taking the space of measurable functions with finite norm $\|u\|_p$ we obtain a space $\widehat{L}^p(\Omega)$. Identifying the functions which differ on a zero measure subset (we say that they are equal a. e. — almost everywhere) we obtain the Lebesgue space $L^p(\Omega)$:

$$L^p(\Omega) = \widehat{L}^p(\Omega) \,\big|_{=\text{a.e.}} = \left\{ u \in \mathcal{M}(\Omega) \,\middle|\, \|u\|_p < \infty \right\} \Big|_{=\text{a.e.}} \, .$$

The spaces $L^p(\Omega)$ are Banach spaces, they are separable except for $p = \infty$ and reflexive except for $p = 1, \infty$.

In case $p = 2$ the space $L^2(\Omega)$ is Hilbert space with scalar product

$$(u, v) = \int_\Omega u \, v \, \mathrm{d}x \, . \tag{12.3}$$

The induced norm coincides with $\| \cdot \|_2$.

There are natural imbeddings for domains Ω of finite measure $\mu(\Omega) < \infty$:

$$L^q(\Omega) \subset L^p(\Omega) \qquad \text{for } \; p < q \, . \tag{12.4}$$

Indeed, (11.8) with $p = q/p$, $p' = q/(q-p)$ and $u = |u|^p$ yields

$$\|u\|_p \le \|u\|_q \cdot [\mu(\Omega)]^{(q-p)/(pq)}$$

which proves the imbedding. Particularly

$$L^2(\Omega) \subset L^1(\Omega) \qquad \|u\|_1 \le \|u\|_2 \cdot [\mu(\Omega)]^{1/2} \, . \tag{12.5}$$

The inequalities (11.6) and (11.7) can be rewritten using the L^p norm:

$$\left| \int_\Omega u \, v \, \mathrm{d}x \right| \le \|u\|_2 \cdot \|v\|_2 \qquad \left| \int_\Omega u \, v \, \mathrm{d}x \right| \le \|u\|_p \cdot \|v\|_{p'} \, . \tag{12.6}$$

Sobolev spaces — norms.

The Sobolev spaces of functions on a domain $\Omega \subset \mathbb{R}^N$ has two parameters: the order of derivatives k ($k = 0, 1, 2, 3, \ldots$) and the power p ($p \in [1, \infty]$). We mention the general case but we shall deal particularly with the case $k = 1$ and $p = 2$.

Using the Lebesgue norm $\|u\|_p$ defined by (12.1), (12.2) Sobolev norms $\|u\|_{k,p}$ can be defined by summing the norms of functions and their derivatives up to order k:

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha u\|_p \,,$$

particularly for $k = 1$:

$$\|u\|_{1,p} = \|u\|_p + \left\| \frac{\partial u}{\partial x_1} \right\|_p + \cdots + \left\| \frac{\partial u}{\partial x_N} \right\|_p \,.$$

Since for $p = 2$ we can introduce a scalar product $(\cdot, \cdot)_k$ and we want equality $(u, u)_k = \|u\|_{k,2}^2$ holds, we introduce another (k, p)-norm which is not equal but equivalent (both give the same convergence and the same topology) to the previous one:

$$\|u\|_{k,p} = \left[\sum_{|\alpha| \leq k} \|\mathrm{D}^\alpha u\|_p^p \right]^{1/p} , \tag{12.7}$$

particularly for $k = 1$ and $p = 2$ we have the norm:

$$\begin{aligned} \|u\|_{1,2} &= \left[\left\| \frac{\partial u}{\partial x_1} \right\|_2^2 + \cdots + \left\| \frac{\partial u}{\partial x_N} \right\|_2^2 + |u|_2^2 \right]^{1/2} \\ &= \left[\int_\Omega \left(\left| \frac{\partial u}{\partial x_1} \right|^2 + \cdots + \left| \frac{\partial u}{\partial x_N} \right|^2 + |u|^2 \right) \mathrm{d}x \right]^{1/2} \end{aligned} \tag{12.8}$$

and the corresponding scalar product:

$$\begin{aligned} (u, v)_1 &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1} \right) + \cdots + \left(\frac{\partial u}{\partial x_N}, \frac{\partial v}{\partial x_N} \right) + (u, v) \\ &= \int_\Omega \left(\frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \cdots + \frac{\partial u}{\partial x_N} \frac{\partial v}{\partial x_N} + u\, v \right) \mathrm{d}x \,. \end{aligned} \tag{12.9}$$

Definitions.

There are three ways of introducing the Sobolev space. The first consists in completion the set of smooth functions, the second in selecting the function from the set of distributions — generalized functions. The third is based on the notion of absolutely continuous functions. We shall suppose that Ω is a bounded domain in $\mathbb{R}^N$.

The first way $W^{k,p}(\Omega)$**:** Let the domain Ω be bounded. We start with the set of all infinitely differentiable functions on $\mathbb{R}^N$ restricted to the domain Ω denoted by $\mathcal{E}(\Omega)$. Then the Sobolev space denoted by $W^{k,p}(\Omega)$ is defined by completion in the corresponding norm:

$$W^{k,p}(\Omega) = \textit{completion of the set } \mathcal{E}(\Omega) \textit{ in the norm } \|\cdot\|_{k,p} \,.$$

The second way $H^{k,p}(\Omega)$**:** Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions with compact support in Ω and let $\mathcal{D}'(\Omega)$ be its dual i. e. the set of all continuous functionals in $\mathcal{D}(\Omega)$. These functionals are called distributions. Let us remind that they are continuous in the following sense: $\langle T, \varphi_n\rangle \to 0$ for any sequence $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ such that all φ_n are zero outside an compact set $K \subset \Omega$ and all derivatives $\mathrm{D}^\alpha \varphi_n$ converge to 0 uniformly on Ω.

In general, the distributions are not functions, nevertheless some of them can be represented by integrable functions. Any function $f \in L^1(\Omega)$ defines a distribution T defined by its value at functions φ from $\mathcal{D}(\Omega)$:

$$\langle T, \varphi\rangle = \int_\Omega f\,\varphi\,\mathrm{d}x \quad \forall\,\varphi \in \mathcal{D}(\Omega)\,.$$

The distributions that can be represented in this form are called *regular*. We identify them with its representing integrable function. In this sense $L^p(\Omega)$ is a subspace of the set of distribution $\mathcal{D}'(\Omega)$.

The distributions form a linear space, they can be multiplied by C^∞ functions and differentiated as many times as one wants by the formula:

$$\langle \mathrm{D}^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\,\langle T, \mathrm{D}^\alpha\varphi\rangle \quad \forall\,\varphi \in \mathcal{D}(\Omega)\,.$$

Thus any distribution T has derivatives of all order in this sense called in the sense of distributions. These derivatives can be again regular (or not) and thus we can decide whether they are in $L^p(\Omega)$ or not. If the usual derivative exists then it coincides with the distributional one.

The Sobolev space denoted by $H^{k,p}(\Omega)$ is *the set of distributions such that their distributional derivatives up to order k can be represented as* L^p*-functions*:

$$H^{k,p}(\Omega) = \left\{u \in \mathcal{D}'(\Omega)\,\middle|\; \mathrm{D}^\alpha u \in L^p(\Omega) \quad \forall\alpha\,, |\alpha| \le k\right\},$$

The third way. Let us briefly mention the Beppo-Levi definition of Sobolev spaces, see [9], the $k = 1$ case only.

Let us denote by $AC_i(\Omega)$ the space of functions absolutely continuous on almost all closed segments in Ω parallel to x_i axis. These functions have derivatives $(\partial u)/(\partial x_i)$ almost everywhere in Ω. Then the Sobolev space with p and

$k = 1$ can be defined by

$$\left\{ u \in \bigcap_i AC_i(\Omega) \,\Big|\, \; u\,, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \in L^p(\Omega) \right\}\,.$$

Comparison of the definitions.

Beppo Levi construction yields the same space $W^{1,p}(\Omega)$ as the first definition. The second definition is different. In general we have only

$$W^{k,p}(\Omega) \subset H^{k,p}(\Omega)\,.$$

In case of *domains with Lipschitz boundary* and $p < \infty$ we obtain the same function spaces:

$$W^{k,p}(\Omega) = H^{k,p}(\Omega) \qquad \text{for } p \in [1, \infty)\,.$$

That is one of the reasons why we shall deal with domains Ω with Lipschitz boundary only. Nevertheless for $p = \infty$ we have $W^{k,\infty}(\Omega) \subset H^{k,\infty}$; particularly $W^{0,\infty}(\Omega) = C^0(\Omega)$ and $H^{0,\infty}(\Omega) = L^\infty(\Omega)$.

Zero trace spaces and equivalent norms.

Starting from the space of smooth functions with compact support in Ω denoted by $C_0^\infty(\Omega)$ we obtain

$$W_0^{k,p}(\Omega) = \textit{completion of } C_0^\infty(\Omega) \textit{ in the norm } \|\cdot\|_{k,p}\,.$$

The functions have zero values and zero derivatives up to order $k-1$ in a generalized sense at the boundary.

This construction is used for spaces for boundary value problems with stable boundary conditions. The boundary condition $u = 0$ on Γ_0 $(\Gamma_0 \subset \partial\Omega)$ attached to a second order differential equation is included into the Sobolev space. The solution is looked for in the space V defined as completion of the set

$$\mathcal{V} = \left\{ u \in \mathcal{E}(\Omega) \,\big|\, \; u = 0 \quad \text{on } \Gamma_0 \right\} \tag{12.10}$$

in the (1,2)-norm, or the closure of the same set in $W^{1,2}(\Omega)$.

The functional norming only the highest order derivatives

$$|u|_{k,p} = \left[\sum_{|\alpha|=k} \|\mathrm{D}^\alpha u\|_p^p \right]^{1/p} \tag{12.11}$$

is a seminorm on $W^{k,p}(\Omega)$ since for $k-1$ order polynomials p we have $|p|_{k,p} = 0$. But on zero trace spaces $W_0^{k,p}(\Omega)$ the introduced seminorm is equivalent to the the norm (12.7). Particularly, the seminorm

$$|u|_{1,2} = \left[\int_\Omega \left(\left| \frac{\partial u}{\partial x_1} \right|^2 + \cdots + \left| \frac{\partial u}{\partial x_N} \right|^2 \right) \mathrm{d}x \right]^{1/2} . \tag{12.12}$$

represents the norm on $W_0^{1,2}(\Omega)$ equivalent to the norm defined by (12.8). It is true even in case of Sobolev spaces defined by completion of the set (12.10) with Γ_0 having positive surface measure.

Properties of the Sobolev spaces.

The Sobolev spaces $W^{k,p}(\Omega)$ are separable in case $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$. For $p = 1$ the space $W^{k,p}(\Omega)$ is not reflexive, for $p = \infty$ it is neither separable nor reflexive. The same holds for its zero trace subspaces.

Theorems on imbeddings.

Besides the natural imbeddings $W^{k,p} \subset W^{l,q}$ for $l \le k$ and $q \le p$ (in case of the bounded domain Ω) there are other imbeddings of the type "for less order of derivatives $l < k$ we can obtain integrability in higher power $q > p$". Proof of each imbedding $X \subset Y$ is enabled by estimate $\|u\|_Y \le \text{const} \cdot \|u\|_X$. We introduce the imbeddings of $W^{1,p}(\Omega)$ and corresponding estimates. The imbedding depends on the dimension N.

Theorem on imbeddings. (see e. g. [9]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $N \ge 2$ and $p \in [1, \infty)$. Then:*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \qquad \|u\|_q \le c \cdot \|u\|_{1,p} \tag{12.13}$$

provided

— *either $p < N$ and $q \in [1, q^*]$ where q^* satisfies $1/q^* = 1/p - 1/N$,*
— *or $p = N$ and $q \in [1, \infty)$ and*

$$W^{1,p}(\Omega) \subset C^0(\overline{\Omega}) \quad [\subset L^\infty(\Omega)] \qquad \max_\Omega |u| \le c \cdot \|u\|_{1,p} , \tag{12.14}$$

provided $p > N$. The constants are independent of u.

Remarks: By multiple application of the theorem we obtain e. g. for $kp > N$ $W^{k,p}(\Omega) \subset C^0(\overline{\Omega})$. In (12.14) we have imbedding of $W^{1,p}(\Omega)$ not only into $C^0(\bar{\Omega})$ but even into the space of Hölder continuous functions $C^{0,\lambda}(\Omega)$.

If the imbedding is not "sharp" we have even compact imbedding:

Theorem on compact imbeddings. (see e. g. [9]) *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with Lipschitz boundary,* $N \geq 2$ *and* $p \in [1, \infty)$. *Then the following imbeddings are compact:*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \tag{12.15}$$

provided

— *either* $p < N$ *and* $q \in [1, q^*)$ *where* q^* *satisfies* $1/q^* = 1/p - 1/N$,

— *or* $p = N$ *and* $q \in [1, \infty)$ *and*

$$W^{1,p}(\Omega) \subset\subset C^0(\overline{\Omega}) \quad [\subset L^\infty(\Omega)] \, , \tag{12.16}$$

provided $p > N$.

The compact imbedding $X \subset\subset Y$ is a strongly continuous identity map $X \to Y$. It means that any set bounded and closed in X is compact in Y; particularly a sequence weakly converging in X is strongly converging in Y.

Theorem on traces.

Integrable functions are defined up to a zero measure subsets. In general, the values on the boundary are not defined, since the boundary has measure zero. In case of Sobolev spaces with $k \geq 1$ the values at the boundary (or a (N-1)-dimensional "good surface") can be determined as a $L^p(\partial\Omega)$ function.

We extend the usual restriction operator giving to a continuous function f on $\overline{\Omega}$ its restriction to $\partial\Omega$ by continuity to a mapping between Sobolev and Lebesgue space: $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$. These restrictions are called **traces**.

Theorem on traces. (see e. g. [9]) *Let* Ω *be a domain with Lipschitz boundary. Then the restriction operator*

$$T : u \in C^\infty(\Omega) \mapsto u|_{\partial\Omega}$$

can be uniquely extended to a continuous operator $\mathcal{T}$ *mapping the spaces*

$$\mathcal{T} : W^{1,p}(\Omega) \to L^p(\partial\Omega) \, .$$

The trace operator is contiuous and linear, thus is bounded, i. e.

$$\|u\|_{L^p(\partial\Omega)} \leq c\|u\|_{1,p} \, , \tag{12.17}$$

where the constant c *depends on* Ω, $\partial\Omega$ *and* p.

13. Nemytskij operators.

The composite operators between Lebesgue spaces can be justified by means of the following theorem:

Theorem on Nemytskij operators.

Let Ω be a domain in $\mathbb{R}^N$ and $h(x,\xi)$ a function

$$h : \Omega \times \mathbb{R}^m \to \mathbb{R}\,.$$

(a) *Let h satisfy the Carathéodory conditions, i.e.*

$$h(x,\xi) \begin{cases} \text{is measurable in } x \text{ for all fixed } \xi \in \mathbb{R}^m, \\ \text{is continuous in } \xi \text{ for almost all } x \in \Omega, \end{cases} \tag{13.1}$$

Then the composed function $h(\cdot, u_1, u_2, \ldots, u_m)$ is measurable for all measurable $u_1, u_2, \ldots, u_m$.

(b) *Let the function h satisfy the Carathéodory conditions and let $p_1, p_2, \ldots, p_m, r \in [1,\infty)$ be given constants.*

Then the Nemytskij operator

$$\mathcal{H} : u_1, u_2, \ldots, u_m \quad \mapsto \quad h(\cdot, u_1(\cdot), u_2(\cdot), \cdots, u_m(\cdot))$$

maps the spaces

$$\mathcal{H} : L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \ldots \times L^{p_m}(\Omega) \quad \to \quad L^r(\Omega) \tag{13.2}$$

if and only if the function h satisfies the following growth condition

$$|h(x,\xi_1,\xi_2,\ldots,\xi_m)| \le g(x) + c\sum_{i=1}^{m} |\xi_i|^{p_i/r}\,, \tag{13.3}$$

where $g \in L^r(\Omega)$ and c is a positive constant.

(b') *The assertion (b) holds even if $p_i = \infty$ for some i. Let $p_i = \infty$ for $i = 1, 2, \ldots, s\,;\ s \le m$ and $p_i, r \in [1,\infty)$ for $i = s+1, \ldots m$.*

Then the assertion (b) *holds if we replace the growth condition by*

$$|h(x,\xi_1,\xi_2,\ldots,\xi_m)| \le c\left(\sum_{i=1}^{s} |\xi_i|\right)\left[g(x) + \sum_{i=s+1}^{m} |\xi_i|^{p_i/r}\right]\,, \tag{13.4}$$

where $g \in L^r(\Omega)$ and $c(t)$ is a continuous positive function.

(b") *In case $r = \infty$ the assertion* (b) *holds if we replace the growth condition by*

$$|h(x,\xi_1,\xi_2,\ldots,\xi_m)| \le c\,, \tag{13.5}$$

where c is a positive constant.

(c) *If the assumptions of* (b) *or* (b') *are satisfied, then the operator $\mathcal{H}$ is continuous and bounded as the mapping of the corresponding function spaces. If the assumptions of* (b") *are satisfied then the operator is bounded (but need not be continuous).*

14. Continuity properties.

In this section we compare various types of continuity. We start with their definitions. We added boundedness since it is equivalent to continuity in case of linear operators.

Definitions. *Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is*

— **continuous** *iff*

$$(C) \qquad u_n \to u \quad \Longrightarrow \quad A(u_n) \to A(u)\,,$$

— **demicontinuous** *iff*

$$(dC) \qquad u_n \to u \quad \Longrightarrow \quad A(u_n) \overset{w}{\rightharpoonup} A(u)\,,$$

— **strongly continuous** *iff*

$$(sC) \qquad u_n \overset{w}{\rightharpoonup} u \quad \Longrightarrow \quad A(u_n) \to A(u)\,,$$

— **weakly continuous** *iff*

$$(wC) \qquad u_n \overset{w}{\rightharpoonup} u \quad \Longrightarrow \quad A(u_n) \overset{w}{\rightharpoonup} A(u)\,,$$

— **hemicontinuous** (i. e. weakly continuous on lines) *iff*

$$(hC) \qquad \{t_n\} \subset \mathbb{R}\,, t_n \to 0 \quad \Longrightarrow \quad A(u + t_n v) \overset{w}{\rightharpoonup} A(u)\,,$$

— **continuous on lines** *iff*

$$(lC) \qquad \{t_n\} \subset \mathbb{R}\,, t_n \to 0 \quad \Longrightarrow \quad A(u + t_n v) \to A(u)\,,$$

— **continuous on finite dimensional subspaces** *iff*

(fC) $\quad A : V_n \to V_n'$ *is continuous for each subspace V_n of finite dimension*

— **Lipschitz continuous** *iff*

$$(LC) \qquad \exists\, L > 0 \;\; s.\,t. \;\; \|A(u_1) - A(u_2)\| \;\le\; L \cdot \|u_1 - u_2\|$$

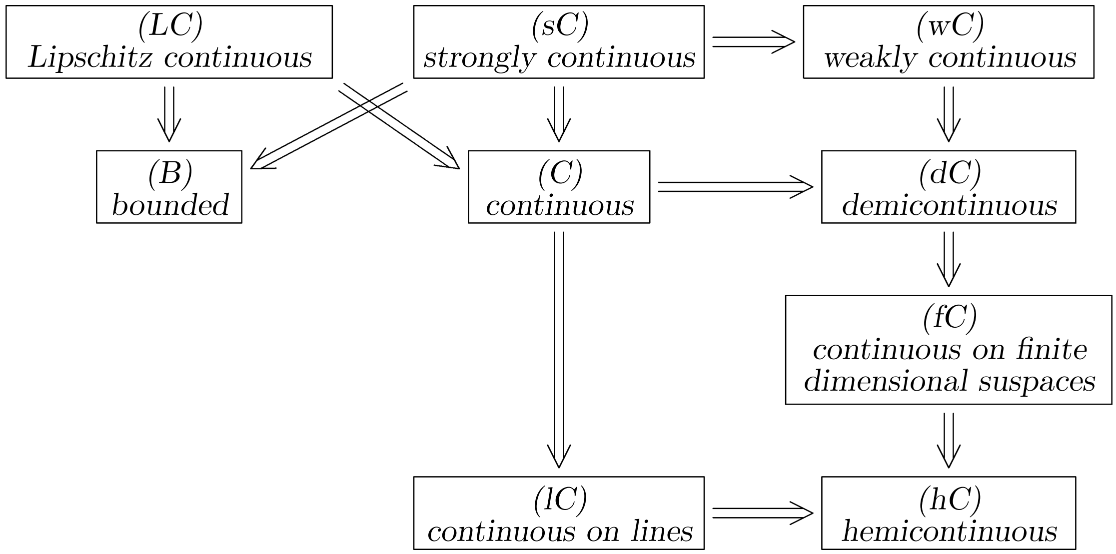
— **bounded** *iff*

(B)

$$M \text{ — bounded in } V \quad \Longrightarrow \quad A(M) = \{A(u) \,\big|\; u \in M\} \text{ — bounded in } V'\,.$$

Remark. Terminology is not entirely unified, sometimes (sC) is called completely continuous.

Lemma 14.1. **(a)** *The following implications holds:*



On the finite dimensional spaces the following concepts coincide:

$$(C) = (sC) = (wC) = (dC) = (fC) \text{ and } (lC) = (hC).$$

For linear operators we have (LC) = (C) = (B).

(b) *The set of operators of each of the introduced continuities forms a linear space, i. e.*

$$A_1, A_2 \in (xC) \quad \Longrightarrow \quad t_1 A_1 + t_2 A_2 \in (xC) \quad \forall\, t_1, t_2 \in \mathbb{R}\,.$$

(c) *The sum of operators of different but comparable continuity forms an operator of the "weaker" continuity.*

The proofs follows immediately from the definitions and from the properties of the strong and weak convergences. In general all definitions define mutually different sets of operators.

15. Monotony properties.

In this section we summarize and compare the monotony like properties including (S)-conditions and coercivity.

Definitions. *Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is*
— **monotone** *iff*

$$(Mon) \qquad \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0 \quad \forall\, u_1, u_2\,,$$

— **strictly monotone** *iff*

(*rMon*) $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle > 0 \quad \forall\, u_1, u_2;\; u_1 \neq u_2\,,$

— **strongly monotone** *iff*

(*sMon*) $\exists\, \alpha > 0 \quad s.\,t. \quad \langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \alpha \cdot \|u_1 - u_2\|^2 \quad \forall\, u_1, u_2\,,$

— **coercive** *iff*

(*coer*)
$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u \rangle}{\|u\|} = \infty,$$

— **satisfies the condition** (S_+) *iff*

(S_+)
$$\Big[u_n \overset{w}{\rightharpoonup} u\,, \quad \limsup \langle A(u_n) - A(u), u_n - u \rangle \leq 0\Big] \quad \Longrightarrow \quad u_n \to u\,,$$

— **satisfies the condition** (S) *iff*

(S)
$$\Big[u_n \overset{w}{\rightharpoonup} u\,, \quad \langle A(u_n) - A(u), u_n - u \rangle \to 0\Big] \quad \Longrightarrow \quad u_n \to u\,,$$

— **satisfies the condition** (S_0) *iff*

(S_0)
$$\Big[u_n \overset{w}{\rightharpoonup} u\,, \quad A(u_n) \overset{w}{\rightharpoonup} b\,, \quad \langle A(u_n), u_n \rangle \to \langle b, u \rangle\Big] \quad \Longrightarrow \quad u_n \to u\,.$$

Remarks.

(a) Monotony properties *(Mon)*, *(sMon)*, *(rMon)* have globalizing character in the following sense: if the inequality holds locally, i.e. for each $u_1, u_2 \in U$, where U are from an open covering of the space V, then the inequality holds for each $u, v \in V$.

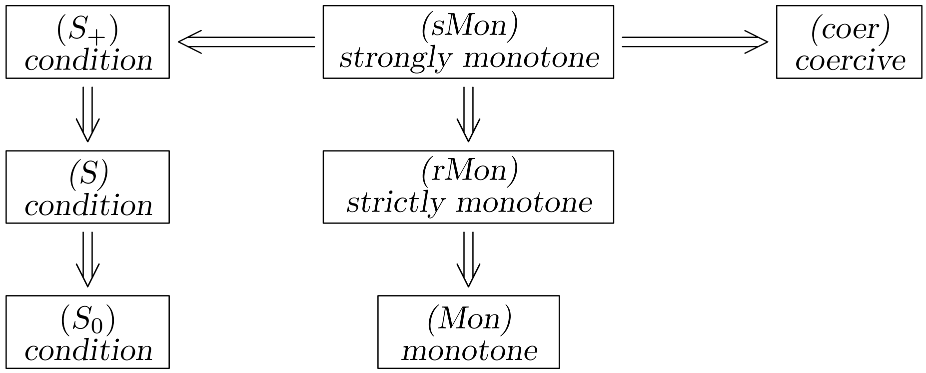
(b) Strongly monotone operators satisfy the implication:

$$\langle A(u_n) - A(u), u_n - u \rangle \to 0 \quad \Longrightarrow \quad u_n \to u\,.$$

The (S)-conditions generalize this property to weakly convergent sequences. They ensure the strong convergence of Galerkin approximation, see Theorem 8.2. In the definitions we replaced $\langle A(u_n) - A(u), u_n - u \rangle$ by $\langle A(u_n), u_n - u \rangle$ since for $u_n \overset{w}{\rightharpoonup} u$ implies $\langle A(u), u_n - u \rangle \to 0$.

(c) Condition (S_+) is used also in theory of Leray-Schauder degree, see [14].

Lemma 15.1. **(a)** *The following implications holds:*



(b) *The sets of operators (Mon), (rMon), (sMon), (coer) form cones, i. e.*

$$A_1, A_2 \in \textit{(xMon)} \quad \Longrightarrow \quad A_1 + A_2 \in \textit{(xMon)}\,, \quad tA_1 \in \textit{(xMon)} \quad t > 0\,.$$

(c) *The sum of two operators of various types of monotony (Mon), (rMon), (sMon) forms an operator of the stronger monotony. Adding an operator (Mon) or (rMon) does not violate coercivity.*

The proof follows from the definitions.

16. Continuity and monotony properties.

The following properties contain both continuity and monotony like properties.

Definitions. *Let $A : V \to V'$ be an operator on a Banach space V. We say that the operator A is*
— **pseudomonotone** *iff*

$$(PM) \qquad \Big[u_n \overset{w}{\rightharpoonup} u\,, \quad \limsup \langle A(u_n), u_n - u\rangle \le 0 \Big] \quad \Longrightarrow$$

$$\Longrightarrow \quad [\, \liminf \langle A(u_n), u_n - v\rangle \ge \langle A(u), u - v\rangle \quad \forall\, v \in V\,]\,,$$

— **satisfies condition** (M) *iff*

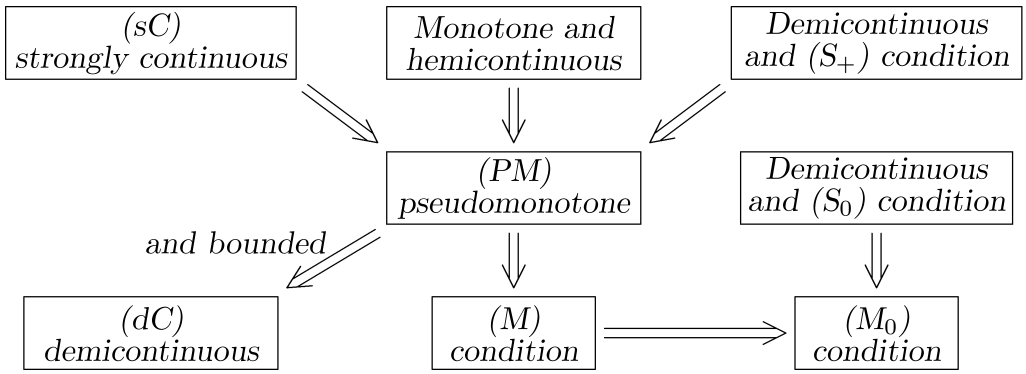
$$(M) \quad \Big[\langle u_n \overset{w}{\rightharpoonup} u\,, \quad A(u_n) \overset{w}{\rightharpoonup} b\,,\ \limsup \langle A(u_n), u_n \le \langle b, u\rangle \Big] \quad \Longrightarrow \quad A(u) = b\,,$$

— **satisfies condition** (M_0) *iff*

$$(M_0) \qquad \Big[\langle u_n \overset{w}{\rightharpoonup} u\,, \quad A(u_n) \overset{w}{\rightharpoonup} b\,,\ \langle A(u_n), u_n \to \langle b, u\rangle \Big] \quad \Longrightarrow \quad A(u) = b\,.$$

Remark. In addition to condition (PM) in the definition of pseudomonotone operators some authors require boundedness or demicontinuity. In the finite dimensional space a continuous operator is psedomonotone and locally bounded pseudomonotone operator is continuous.

Lemma 16.1. *Let $A : V \to V'$ be an operator on a reflexive Banach space V. Then the following implications hold:*



For proofs see [1], Lemma 6.7.

Lemma 16.2..

(a) *The sum of two pseudomonotone operators is a pseudomonotone operator, i. e. the pseudomonotone operators form a cone.*

(b) *The sum of two operators satisfying (S_+) is an operator satisfying (S_+), i. e. the operators satisfying condition (S_+) form a cone.*

(c) *Adding a strongly continuous operator does not violate the property (S_+), (S), (S_0), (PM), (M), (M_0).*

The proofs are based on verifying that if the sum $A_1 + A_2$ satifies the assumption of the property under consideration then both operators satisfy it as well. For details see [1].

17. Weakly continuous operators.

We introduce some propositions which help verification of the assumption that the operator is weakly contiuous. We formulate some results for second order operators only. For general case see [2].

Lemma 17.1. *The set of weakly continuous operators $A : V \to V'$ forms a linear space, i.e. if A_1, A_2 are weakly continuous and $c \in R$ then $A_1 + A_2$ and $c \cdot A_1$ is also weakly continuous.*

The assertion is a simple consequence of definitions. It was already stated in Lemma 14.1 (b). It allows to split operator into a sum of operators and verify their weak continuity separately.

Lemma 17.2. *The linear continuous operator on a reflexive Banach space is weakly continuous.*

Proof. Let $A : V \to V'$ be a linear continuous operator and let $u_n \overset{w}{\rightharpoonup} u$. Let us introduce the adjoint operator to A, i. e. operator $A^* : V \to V'$ defined by

$$\langle A^*(v), u\rangle = \langle A(u), v\rangle \qquad \forall\, u, v \in V\,.$$

Obviously A^* is also continuous. Then for any $v \in V$ we have $A^*(v) \in V'$ and $u_n \overset{w}{\rightharpoonup} u$ implies $A(u_n) \overset{w}{\rightharpoonup} A(u)$ since

$$\langle A(u_n) - A(u), v\rangle = \langle A^* v, u_n - u\rangle \to 0\,.$$

□

Differential operators.

In general for nonlinear operators we can only say: *the strongly continuous operator is weakly continuous*, see Lemma 16.1. In case of differential operators for boundary value problems we can say more: quasilinear (i. e. operators linear in the highest derivatives of the unknowns) are weakly continuous.

Let Ω be a bounded domain in R^N with Lipschitz boundary. We shall deal with second order operators $A : V \to V'$ on closed subspaces V of the Sobolev space $W^{1,2}(\Omega)$ of the following types:

$$\langle A(u), v\rangle = \int_\Omega \frac{\partial u}{\partial x_j} h(\cdot, u) \frac{\partial v}{\partial x_i}\, \mathrm{d}x\,, \tag{17.1}$$

$$\langle A(u), v\rangle = \int_\Omega \frac{\partial u}{\partial x_j} h(\cdot, u) v\, \mathrm{d}x\,, \tag{17.2}$$

$$\langle A(u), v\rangle = \int_\Omega h(\cdot, u) \frac{\partial v}{\partial x_i}\, \mathrm{d}x\,, \tag{17.3}$$

$$\langle A(u), v\rangle = \int_\Omega h(\cdot, u) v\, \mathrm{d}x\,, \tag{17.4}$$

Lemma 17.3. *Let $W^{1,2}(\Omega) \subset\subset L^q(\Omega)$ be the compact imbedding, see Theorem on compact imbeddings, Section 12. Let the function $h(x, \xi)$ satisfy Carathéodory condition (13.1) and the growth conditions:*

(a)

$$|h(x, \xi)| \le \text{const.}$$

in case of operator of type (17.1),

(b)

$$|h(x,\xi)| \le g(x) + c|\xi|^{q/r},$$

where $g \in L^r(\Omega)$, c *a constant and*

$r = 2q/(q-2)$ *in case of operator of type (17.2),*

$r = 2$ *in case of operator of type (17.3),*

$r = q/(q-2)$ *in case of operator of type (17.4).*

In the one-dimesional case $N = 1$ *we have* $q = \infty$ *and the growth condition reads* $|h(x,\xi)| \le c(|\xi|)$, *where* $c(t)$ *is a continuous function.*

Then the operator A *defined by (17.1)–(17.4) is weakly continuous.*

Proof. The proof is based on the Theorem on Nemytskij operators and properties of the weak convergence. The Carathéodory condition with growth condition ensures that the expression $\langle A(u), v\rangle$ is well defined. Let $v \in V$ and $u_n \overset{w}{\rightharpoonup} u$ in $W^{1,2}(\Omega)$. Then

$$\frac{\partial u_n}{\partial x_j} \overset{w}{\rightharpoonup} \frac{\partial u}{\partial x_j} \quad \text{weakly in } L^2(\Omega)\ , \qquad u_n \to u \quad \text{strongly in } L^q(\Omega)$$

and the result follows in case of operators of type (17.3) and (17.4).

In the other cases we split the expression into two parts: the first with the strongly converging sequence and the second with the weakly converging sequence e. g.

$$\langle A(u_n) - A(u), v\rangle = \int_\Omega \frac{\partial u_n}{\partial x_j}\,[h(u_n) - h(u)]\,\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \int_\Omega \left[\frac{\partial u_n}{\partial x_j} - \frac{\partial u}{\partial x_j}\right] h(u) \frac{\partial v}{\partial x_i}\mathrm{d}x$$

and the result follows since both integrals tend to zero. □

In [2], Proposition 2.5 and 2.6 the weak continuity is proved in more general case. Let V be a subspace of $W^{k,p}(\Omega)$ and $A : V \to V'$ be a quasilinear divergent form operator defined by

$$\langle A(u), v\rangle = \int_\Omega \mathrm{D}^{\alpha_0}u \cdot h(\cdot, \mathrm{D}^{\alpha_1}u, \mathrm{D}^{\alpha_2}u, \ldots, \mathrm{D}^{\alpha_m}u)\, \mathrm{D}^{\beta}v\, \mathrm{d}x\,,$$

and lower order operators

$$\langle A(u), v\rangle = \int_\Omega h(\cdot, \mathrm{D}^{\alpha_1}u, \mathrm{D}^{\alpha_2}u, \ldots, \mathrm{D}^{\alpha_m}u)\, \mathrm{D}^{\beta}v\, \mathrm{d}x\,,$$

where D^{α_0} means the k-th order partial derivatives, $\mathrm{D}^{\alpha_1}, \ldots, \mathrm{D}^{\alpha_m}$ lower order derivatives and D^{β} a derivative of order less or equal k.

The linear mappings $u \to \mathrm{D}^{\alpha_i}u$ are supposed to be compact as mappings $W^{k,p}(\Omega) \to L^{p_i}(\Omega)$ for $i = 1, \ldots, m$. The function $h(x, \xi_1, \ldots, \xi_m)$ is supposed to satisfy the Carathéodory conditions and corresponding growth conditions derived from (13.3)–(13.5) such that the integral $\langle A(u), v\rangle$ is "well defined".

The assertion of the result is that the operator A is weakly continuous.

Historical remarks.

The assumption of monotony for operators in a Hilbert space was used by M. Golomb already in 1935. The term "monotone mapping" was invented by R. I. Kačurovskij (1960), who also noticed that the differential of a convex functional is a monotone mapping. The surjectivity of a continuous coercive monotone operator was proved by G. J. Minty (1962) and F. E. Browder (1963). Pseudomonotone operators were introduced by H. Brézis (1968) and F. E. Browder (1968), the operators satisfying condition (M) also by Brézis. Hundreds of papers have been devoted to the theory of monotone operators and its application, more than 300 items are quoted in the monograph [11] (which has been also the main source for these remarks), further references can be found in [12], [4], [5], [6], [7], [10].

The text is a renewed survey paper [1], where more details can be found, completed by [2]. Some useful inequalities and a brief survey of function spaces was added.

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