Aproximace funkcí

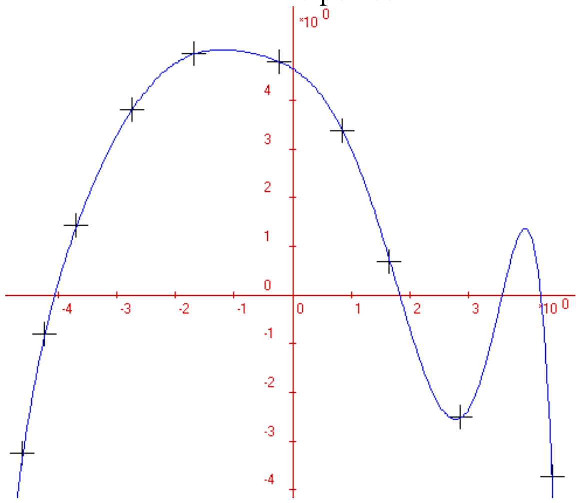
Aproximace = náhrada

Např:

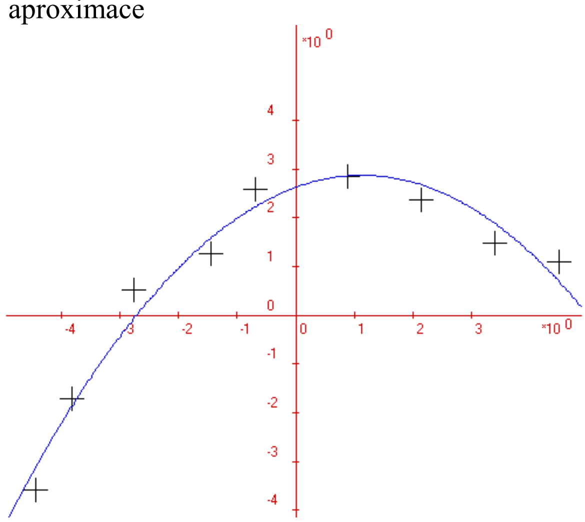
$$\sin x = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - .... + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Nejčastěji:

Interpolace

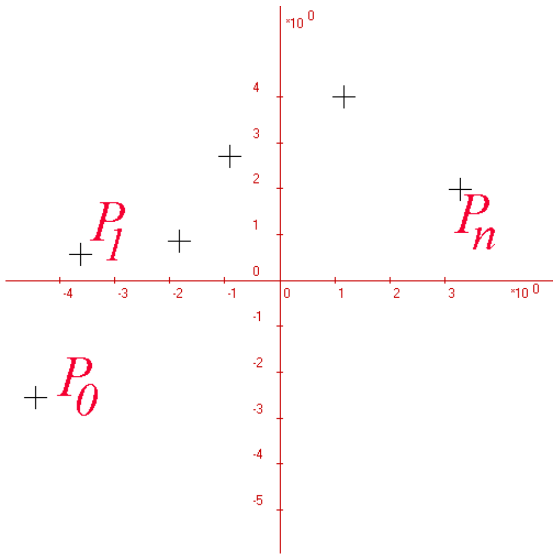


aproximace



Interpolace funkcí

Interpolace polynomem



P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}	....	y_n

Metoda neurčitých koeficientů:

$$y = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$$

$$P_i = [x_i; y_i]; \quad i = 0; 1; \ldots; n$$

$$y_0 = c_n x_0^n + c_{n-1} x_0^{n-1} + .. + c_1 x_0 + c_0$$
$$y_1 = c_n x_1^n + c_{n-1} x_1^{n-1} + .. + c_1 x_1 + c_0$$
....
$$y_n = c_n x_n^n + c_{n-1} x_n^{n-1} + .. + c_1 x_n + c_0$$

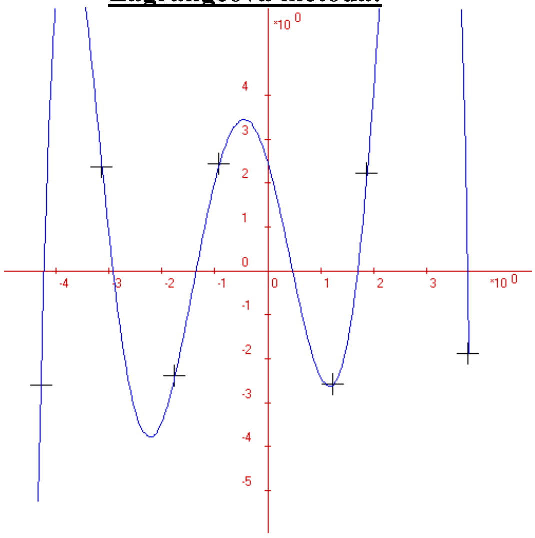
Příklad

Interpolace funkcí

Interpolace polynomem

P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}	....	y_n

Lagrangeova metoda:



$$\ell_i = a(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)$$
$$\ell_i(x_i) = 1$$

$$a(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n) = 1$$

$$a = \frac{1}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

$$\ell_i = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

$$m_i = y_i \ell_i$$

$$L_n(x) = m_0 + m_1 + \dots + m_i + \dots + m_n$$

$$L_n(x) = y_0\ell_0 + y_1\ell_1 + \dots + y_i\ell_i + \dots + y_n\ell_n$$

$$\boxed{L_n(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}}$$

Interpolace funkcí

Lagrangeův interpolační polynom

P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}	....	y_n

$$\boxed{P(x) = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}}$$

Příklad: Pomocí Lagrangeova interpolačního polynomu určete hodnotu $\sqrt{5}$.

Řešení: $y = \sqrt{x}$

x_i	1	4	9
y_i	1	2	3

$$\sqrt{x} \approx L_2(x) = 1 \cdot \frac{(x-4)(x-9)}{(1-4)(1-9)} + 2 \cdot \frac{(x-1)(x-9)}{(4-1)(4-9)} + 3 \cdot \frac{(x-1)(x-4)}{(9-1)(9-4)}$$

$$= \frac{1}{24} \cdot (x-4)(x-9) - \frac{2}{15} \cdot (x-1)(x-9) + \frac{3}{40} \cdot (x-1)(x-4)$$

$$\sqrt{5} \approx L_2(5) = \frac{1}{24} \cdot (5-4) \cdot (5-9) - \frac{2}{15} \cdot (5-1) \cdot (5-9) + \frac{3}{40} \cdot (5-1) \cdot (5-4) = \boxed{2.26} \qquad \sqrt{5} \approx 2.236...$$

Srovnej zde

Newtonova metoda - Newtonův interpolační polynom

P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}	....	y_n

$$N_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Příklad:

	P_0	P_1	P_2
x_i	1	4	9
y_i	1	2	3

$$\sqrt{x} \approx L_2(x) = 1\cdot\frac{(x-4)(x-9)}{(1-4)(1-9)} + 2\cdot\frac{(x-1)(x-9)}{(4-1)(4-9)} + 3\cdot\frac{(x-1)(x-4)}{(9-1)(9-4)}$$

$$= \frac{1}{24}\cdot(5-4)\cdot(5-9) - \frac{2}{15}\cdot(5-1)\cdot(5-9) + \frac{3}{40}\cdot(5-1)\cdot(5-4) = \boxed{2.26}$$

$$N_0(x) = a_0 \Rightarrow a_0 = 1$$

$$N_1(x) = 1 + \frac{1}{3}(x-1)$$

$$N_2(x) = 1 + \frac{1}{3}(x-1) + a_2(x-1)(x-4)$$

$$N_2(x) = 1 + \frac{1}{3}(9-1) + a_2(9-1)(9-4) = 3 \Rightarrow a_2 = -\frac{1}{60}$$

$$N_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{60}(x-1)(x-4) \qquad \sqrt{5} \approx N_2(5) = 1 + \frac{1}{3}\cdot(5-1) - \frac{1}{60}\cdot(5-1)\cdot(5-4) = \boxed{2.26}$$

Newtonova metoda - Newtonův interpolační polynom

P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}	....	y_n

$$N_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Příklad:

	P_0	P_1	P_2
x_i	1	4	9
y_i	1	2	3

$$\sqrt{x} \approx L_2(x) = 1\cdot\frac{(x-4)(x-9)}{(1-4)(1-9)} + 2\cdot\frac{(x-1)(x-9)}{(4-1)(4-9)} + 3\cdot\frac{(x-1)(x-4)}{(9-1)(9-4)}$$

$$= \frac{1}{24}\cdot(5-4)\cdot(5-9) - \frac{2}{15}\cdot(5-1)\cdot(5-9) + \frac{3}{40}\cdot(5-1)\cdot(5-4) = \boxed{2.26}$$

$$N_3(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{60}(x-1)(x-4) + \frac{1}{60}(x-1)(x-4)(x-9)$$

ještě jednou:

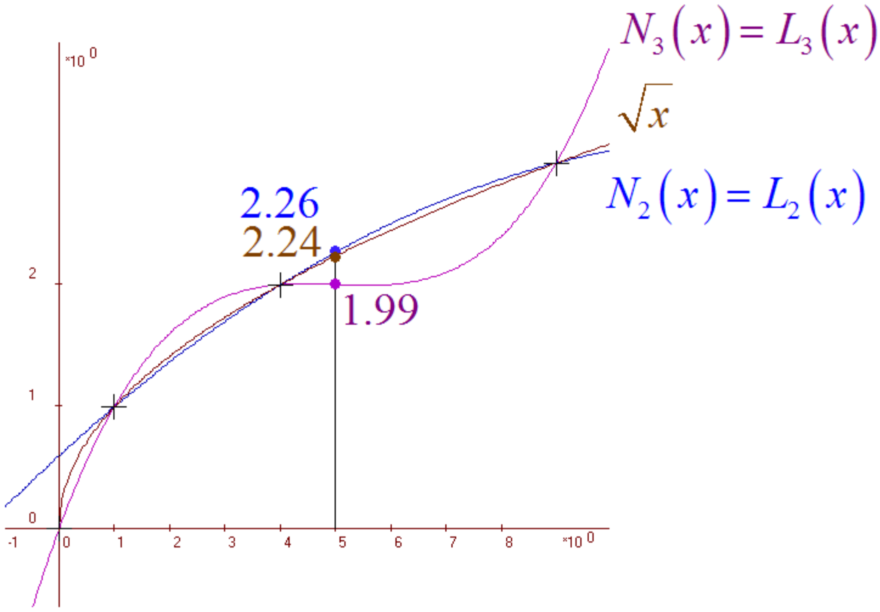
$$\begin{array}{ccccc}
1 & 1 & & & \\
4 & 2 & \frac{1}{3} & & \\
9 & 3 & \frac{1}{5} & -\frac{1}{60} & \\
0 & 0 & \frac{1}{3} & -\frac{1}{30} & \frac{1}{60}
\end{array}$$

$$N3(5) = \underbrace{1 + \frac{1}{3}(5-1) - \frac{1}{60}(5-1)(5-4)}_{2.26} + \underbrace{\frac{1}{60}(5-1)(5-4)(5-9)}_{\approx^{\circ}-0.27} \approx 1.99$$

Interpolační polynom

$$N_2(x)=1+\frac{1}{3}(x-1)-\frac{1}{60}(x-1)(x-4)$$

$$N_3(x)=1+\frac{1}{3}(x-1)-\frac{1}{60}(x-1)(x-4)+\frac{1}{60}(x-1)(x-4)(x-9)$$



Interpolace funkcí

Chyba interpolace interpolačním polynomem

Interpolujeme-li funkci $f(x)$, jejíž hodnoty v uzlových bodech jsou

P_0	P_1	P_2	....	P_i	P_{i+1}	....	P_n
x_0	x_1	x_2	....	x_i	x_{i+1}	....	x_n
y_0	y_1	y_2	....	y_i	y_{i+1}	....	y_n

a $\langle a;b\rangle$ interval takový, že $x_0;x_1;...;x_n\in\langle a;b\rangle$. Pro chybu interpolace polynomem $L_n(x)$ pak platí:

$$E_n(x)\le\frac{1}{(n+1)!}\cdot\max_{x\in\langle a;b\rangle}\left|f^{(n+1)}(x)\cdot(x-x_0)\cdot(x-x_1)\cdot...\cdot(x-x_n)\right|$$

Příklad: Určeme chybu hodnoty $\sqrt{5}\approx 2.26$ stanovené LIP

x_i	1	4	9
y_i	1	2	3

Řešení: $f^{(n+1)}(x)=\left[\sqrt{x}\right]'''=\left[\frac{1}{2}x^{-\frac{1}{2}}\right]''=\left[-\frac{1}{4}x^{-\frac{3}{2}}\right]'=\frac{3}{8}x^{-\frac{5}{2}}=\frac{3}{8x^2\sqrt{x}}$; $\max_{x\in\langle a;b\rangle}\left|f^{(n+1)}(x)\right|=\frac{3}{8}$

Odhad $\max_{x\in\langle a;b\rangle}\left|(x-x_0)\cdot(x-x_1)\cdot...\cdot(x-x_n)\right|$: $x=2.5\Rightarrow(x-1)\cdot(x-4)\cdot(x-9)=1.5\cdot1.5\cdot6.5\approx15$

$x=6.5\Rightarrow(x-1)\cdot(x-4)\cdot(x-9)=5.5\cdot2.5\cdot2.5\approx34$

$$E_n(x)\le\frac{1}{(n+1)!}\cdot\max_{x\in\langle a;b\rangle}\left|f^{(n+1)}(x)\cdot(x-x_0)\cdot(x-x_1)\cdot...\cdot(x-x_n)\right|=\frac{1}{3!}\cdot\frac{3}{8}\cdot34\approx2.1$$

Interpolace funkcí

Hermitův interpolační polynom

Příklad: Pomocí Hermitova interpolačního polynomu určete hodnotu $\sqrt{5}$.

Řešení: $y=\sqrt{x}$

x_i	1	4	9
y_i	1	2	3
y'_i	$\frac{1}{2}$	-	$\frac{1}{6}$

Řešíme tuto soustavu

$$a_4=-0.00045$$
$$a_3=\ \ 0.00983$$
$$a_2=-0.08097$$
$$a_1=0.5;\quad a_0=1$$

Metoda neurčitých koeficientů:

$$\cancel{H_4(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}$$

$$H_4(x)=a_4\left(x-1\right)^4+a_3\left(x-1\right)^3+a_2\left(x-1\right)^2+a_1\left(x-1\right)+a_0$$

$$\frac{d}{dx}H_4(x)=4a_4\left(x-1\right)^3+3a_3\left(x-1\right)^2+2a_2\left(x-1\right)+a_1$$

$$\left.\begin{array}{l}
1=a_4\cdot\left(1-1\right)^4+a_3\cdot\left(1-1\right)^3+a_2\cdot\left(1-1\right)^2+a_1\cdot\left(1-1\right)+a_0\\
2=a_4\cdot\left(4-1\right)^4+a_3\cdot\left(4-1\right)^3+a_2\cdot\left(4-1\right)^2+a_1\cdot\left(4-1\right)+a_0\\
3=a_4\cdot\left(9-1\right)^4+a_3\cdot\left(9-1\right)^3+a_2\cdot\left(9-1\right)^2+a_1\cdot\left(9-1\right)+a_0\\
\tfrac{1}{2}=4a_4\cdot\left(1-1\right)^3+3a_3\cdot\left(1-1\right)^2+2a_2\cdot\left(1-1\right)+a_1\\
\tfrac{1}{6}=4a_4\cdot\left(9-1\right)^3+3a_3\cdot\left(9-1\right)^2+2a_2\cdot\left(9-1\right)+a_1
\end{array}\right\}\Rightarrow
\left.\begin{array}{l}
a_0=1\\
a_1=\frac{1}{2}
\end{array}\right\}\Rightarrow
\begin{array}{r}
2=81a_4+27a_3+9a_2+1.5+1\\
3=4096a_4+512a_3+64a_2+4+1\\
0.1\overline{6}=2048a_4+192a_3+16a_2+0.5
\end{array}$$

Interpolace funkcí

Hermitův interpolační polynom

Příklad: Pomocí Hermitova interpolačního polynomu určete hodnotu $\sqrt{5}$.

Řešení: $y=\sqrt{x}$

$$H_4(x)=a_4\left(x-1\right)^4+a_3\left(x-1\right)^3+a_2\left(x-1\right)^2+a_1\left(x-1\right)+a_0$$

x_i	1	4	9
y_i	1	2	3
y'_i	$\frac{1}{2}$	-	$\frac{1}{6}$

$$a_4=-0.00045$$
$$a_3=\ \ 0.00983$$
$$a_2=-0.08097$$
$$a_1=0.5;$$
$$a_0=1$$

$$H_4(x)=-0.00045\cdot\left(x-1\right)^4+0.00983\cdot\left(x-1\right)^3-0.08097\cdot\left(x-1\right)^2+0.5\cdot\left(x-1\right)+1$$

$$H_4(5)=-0.00045\cdot\left(5-1\right)^4+0.00983\cdot\left(5-1\right)^3-0.08097\cdot\left(5-1\right)^2+0.5\cdot\left(5-1\right)+1\approx\boxed{2.23}$$

$$\sqrt{5}\approx 2.236...$$

$$L_2(5)\approx\boxed{2.26}$$

Interpolace funkcí

Hermitův interpolační polynom Příklad

Příklad: Pomocí Hermitova interpolačního polynomu určete hodnotu $\sqrt{5}$.

Řešení: $y=\sqrt{x}$

$$H_5(x)=a_5\left(x-1\right)^5+a_4\left(x-1\right)^4+a_3\left(x-1\right)^3+a_2\left(x-1\right)^2+a_1\left(x-1\right)+a_0$$

$$\frac{d}{dx}H_5(x)=5a_5\left(x-1\right)^4+4a_4\left(x-1\right)^3+3a_3\left(x-1\right)^2+2a_2\left(x-1\right)+a_1$$

x_i	1	4	9
y_i	1	2	3
y'_i	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$

$$1=a_5\cdot\left(1-1\right)^5+a_4\cdot\left(1-1\right)^4+a_3\cdot\left(1-1\right)^3+a_2\cdot\left(1-1\right)^2+a_1\cdot\left(1-1\right)+a_0$$
$$2=a_5\cdot\left(4-1\right)^5+a_4\cdot\left(4-1\right)^4+a_3\cdot\left(4-1\right)^3+a_2\cdot\left(4-1\right)^2+a_1\cdot\left(4-1\right)+a_0$$
$$3=a_5\cdot\left(9-1\right)^5+a_4\cdot\left(9-1\right)^4+a_3\cdot\left(9-1\right)^3+a_2\cdot\left(9-1\right)^2+a_1\cdot\left(9-1\right)+a_0$$
$$\tfrac{1}{2}=5a_5\cdot\left(1-1\right)^4+4a_4\cdot\left(1-1\right)^3+3a_3\cdot\left(1-1\right)^2+2a_2\cdot\left(1-1\right)+a_1$$
$$\tfrac{1}{4}=5a_5\cdot\left(4-1\right)^4+4a_4\cdot\left(4-1\right)^3+3a_3\cdot\left(4-1\right)^2+2a_2\cdot\left(4-1\right)+a_1$$
$$\tfrac{1}{6}=4a_4\cdot\left(9-1\right)^3+3a_3\cdot\left(9-1\right)^2+2a_2\cdot\left(9-1\right)+a_1$$

$$a_0=1;a_1=\frac{1}{2}$$

$$2=\quad 243\cdot a_5+\quad 81\cdot a_4+\ 27\cdot a_3+\ 9\cdot a_2+\tfrac{3}{2}+1$$
$$3=32\ 768\cdot a_5+4\ 096\cdot a_4+512\cdot a_3+64\cdot a_2+4+1$$
$$\tfrac{1}{4}=\quad 405\cdot a_5+\ 108\cdot a_4+\ 27\cdot a_3+\ 6\cdot a_2+\tfrac{1}{2}$$
$$\tfrac{1}{6}=20\ 480\cdot a_5+2\ 048\cdot a_4+192\cdot a_3+\ 16\cdot a_2+\tfrac{1}{2}$$

$a_5=0{,}0000\ 856;\ a_4=-0{,}0020\ 787;\ a_3=0{,}019\ 419;\ a_2=-0.097\ 417;\ a_1=0{,}5;\ \ a_5=1$; $\quad\sqrt{5}\approx 2.236...$

$H_5(x)=0{,}0000\ 856\cdot x^5-0{,}0020\ 787\cdot x^4+0{,}019\ 419\cdot x^3-0.097\ 417\cdot x^2+0{,}5\cdot x+1$; $\quad H_5(5)=2{,}239...$

Interpolace funkcí

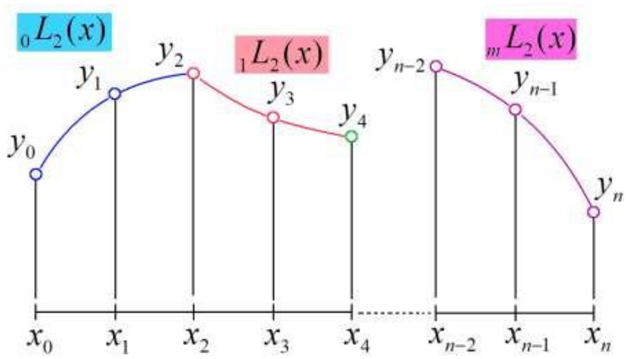
P_0	P_1	....	P_{i-1}	P_i	P_{i+1}	....	P_n
x_0	x_1	....	x_{i-1}	x_i	x_{i+1}	....	x_n
y_0	y_1	....	y_{i-1}	y_i	y_{i+1}		y_n

Globální interpolace

$$\boxed{L_n(x)=\sum_{i=0}^{n}y_i\frac{\left(x-x_0\right)\left(x-x_1\right)...\left(x-x_{i-1}\right)\left(x-x_{i+1}\right)....\left(x-x_n\right)}{\left(x_i-x_0\right)\left(x_i-x_1\right)...\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right)....\left(x_i-x_n\right)}}$$

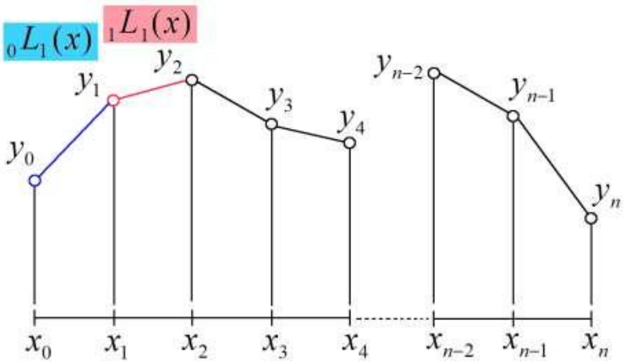
Interpolace po částech - např. kvadratická:

P_{k-1}	P_k	P_{k+1}
x_{k-1}	x_k	x_{k+1}
y_{k-1}	y_k	y_{k+1}



$$L_2(x)=y_{k-1}\frac{\left(x-x_k\right)\left(x-x_{k+1}\right)}{\left(x_{k-1}-x_k\right)\left(x_{k-1}-x_{k+1}\right)}+y_k\frac{\left(x-x_{k-1}\right)\left(x-x_{k+1}\right)}{\left(x_k-x_{k-1}\right)\left(x_k-x_{k+1}\right)}+y_{k+1}\frac{\left(x-x_{k-1}\right)\left(x-x_k\right)}{\left(x_{k+1}-x_{k-1}\right)\left(x_{k+1}-x_k\right)}$$

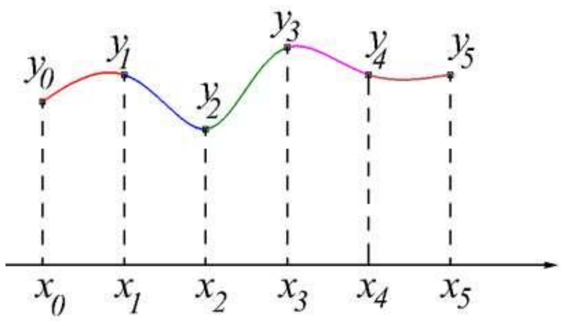
Interpolace po částech - lineární:



P_{k-1}	P_k
x_{k-1}	x_k
y_{k-1}	y_k

$$L_1(x) = y_{k-1}\frac{x - x_k}{x_{k-1} - x_k} + y_k\frac{x - x_{k-1}}{x_k - x_{k-1}}$$

Interpolace po částech - splajny



Splajn $s(x)$ n-tého stupně:

1) Pro každé $i = 0; 1; \dots; n$ je $s(x_i) = y_i$

2) Na každém subintervalu $\langle x_i; x_{i+1}\rangle$; $i = 0; 1; \dots; n-1$ polynom n-tého stupně

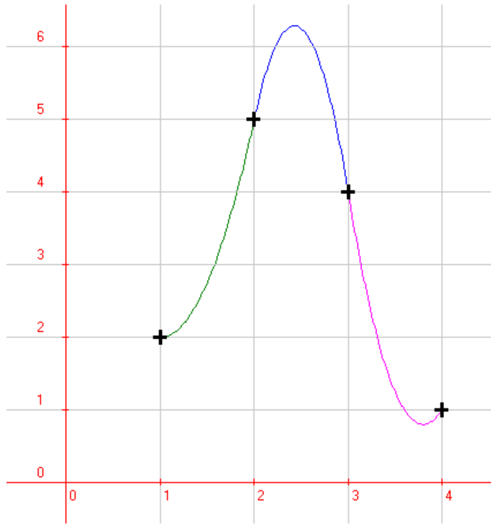
3) Funkce $s(x)$ je na celém intervalu $\langle x_0; x_n\rangle$ spojitá i se svými derivacemi až do řádu $n-1$.

Interpolace po částech - splajny

Příklad:

x_i	1	2	3	4
y_i	2	5	4	1

splajn 2. stupně



$$f_1(x) = a_1x^2 + b_1x + c_1$$

$$f_2(x) = a_2x^2 + b_2x + c_2$$

$$f_3(x) = a_3x^2 + b_3x + c_3$$

$$2 = 1^2 \cdot a_1 + 1 \cdot b_1 + c_1$$
$$5 = 2^2 \cdot a_1 + 2 \cdot b_1 + c_1$$
$$5 = 2^2 \cdot a_2 + 2 \cdot b_2 + c_2$$
$$4 = 3^2 \cdot a_2 + 3 \cdot b_2 + c_2$$
$$4 = 3^2 \cdot a_3 + 3 \cdot b_3 + c_3$$
$$1 = 4^2 \cdot a_3 + 4 \cdot b_3 + c_3$$

$$2a_1x + b_1 = 2a_2x + b_2 \qquad 4a_1 + b_1 = 4a_2 + b_2$$
$$2a_2x + b_2 = 2a_2x + b_2 \qquad 4a_2 + b_2 = 4a_3 + b_3$$
$$f'_1(x) = 2a_1x + b_1;$$
$$f'_1(1) = 0 \qquad 0 = 2a_1 + b_1$$

$a_1 = 3;\quad b_1 = -6;\quad c_1 = 5;$
$a_2 = -7;\quad b_2 = 34;\quad c_2 = -35;$
$a_3 = 5;\quad b_3 = -38;\quad c_3 = 73;$

$$f_1(x) = 3x^2 - 6x + 5$$
$$f_2(x) = -7x^2 + 34x - 35$$
$$f_3(x) = 5x^2 - 38x + 73$$

Interpolace po částech - splajny

Splajn 3. stupně (kubický splajn)

Příklad:

Neznámé $M_1; M_2; \ldots; M_{n-1}$ (druhé derivace v uzlových bodech)
h_i – vzdálenosti mezi uzlovými body

$$\begin{pmatrix} 2(h_1+h_2) & h_2 & 0 & \ldots & 0 \\ h_2 & 2(h_2+h_3) & h_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2(h_{n-1}+h_n) \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ \ldots \\ M_{n-1} \end{pmatrix} = 6 \begin{pmatrix} \frac{1}{h_1}y_0 - \left(\frac{1}{h_1}+\frac{1}{h_2}\right)y_1 + \frac{1}{h_2}y_2 \\ \frac{1}{h_2}y_1 - \left(\frac{1}{h_2}+\frac{1}{h_3}\right)y_2 + \frac{1}{h_3}y_3 \\ \ldots\ldots\ldots\ldots \\ \frac{1}{h_{n-1}}y_{n-2} - \left(\frac{1}{h_{n-1}}+\frac{1}{h_n}\right)y_{n-1} + \frac{1}{h_n}y_n \end{pmatrix}$$

Volíme $M_0; M_n$: $\qquad M_0 = M_n = 0$:

$$s(x) = M_{k-1}\frac{(x_k - x)^3}{6h_k} + M_k\frac{(x - x_{k-1})^3}{6h_k} + \left(y_{k-1} - \frac{M_{k-1}h_k^2}{6}\right)\frac{x_k - x}{h_k} + \left(y_k - \frac{M_k h_k^2}{6}\right)\frac{x - x_{k-1}}{h_k}\ ; \qquad k = 1, \ldots, n$$

Odvození