Brno University of Technology, Faculty of Mechanical Engineering

Modern Methods of Solving Differential Equations

Requirements to Exam for 5th year

study branch Mathematical Engineering – 2018/19

I. Function spaces

1. **Basic notions**
   (a) Metric, linear, normed a unitary spaces, Banach and Hilbert spaces.
   (b) Continuous functional and operators (linear and nonlinear).
   (c) Banach Theorem on Fixed Point of Contractive Mapping.
2. **Lebesgue spaces**
   (a) Norms, properties of $L^p(\Omega)$, space $L^\infty(\Omega)$, inequalities.
   (b) Continuous linear functionals and their representation.
   (c) Separability and reflexivity, weak and strong convergence.
3. **Generalized functions – distributions**
   (a) Basic space, convergence, distribution.
   (b) Regular and singular distributions, examples.
   (c) Operations on distributions, derivatives in sense of distributions.
4. **Description of domain boundary**
   (a) Domain with Lipschitz boundary; definition and examples.
   (b) Domain with inner and outer cone property.
5. **Sobolev spaces**
   (a) Spaces $W^{k,p}(\Omega)$, $H^{k,p}(\Omega)$, $BL^p(\Omega)$ and their comparison.
   (b) Spaces $W_0^{1,p}(\Omega)$, equivalent norms.
   (c) Embedding and compact embedding of spaces $W^{1,p}(\Omega)$.
   (d) Theorem on traces of $W^{1,p}(\Omega)$, idea of the proof.
   (e) Properties: complete, separable and reflexive spaces.
   (f) Continuous linear functionals and their representation.
   (g) Weak and strong convergence.
   (h) Case $p = 2$ – Hilbert space, inner product,
   continuous linear functionals and their representation.

II. Stationary linear problems

1. **Weak formulation of problems**
   (a) Weak and classical formulation and their relation.
   (b) Justification of the weak formulation.
   (c) Case of discontinuous coefficients, transmission conditions.
   (d) Lax-Milgram Lemma, its proof.
   (e) Existence, uniqueness and estimate of solutions to elliptic equations.
2. **Variational formulation a approximation**
   (a) Variational formulation, existence a uniqueness of solutions.
   (b) Relation between weak and variational formulations, proof.
   (c) Galerkin and Ritz approximation.
   (d) Convergence of finite dimensional approximations.
   (e) Finite element method, its advantage.

III. Stationary nonlinear problems

1. **Linear and nonlinear problems** — comparison, various nonlinearities.
2. **Nemytskii operators** — measurability, integrability , boundedness and continuity of operators.
3. **Problems in variational formulation**
   (a) Abstract variational problem: formulation and theorem on existence, proof.
   (b) Justification of variational formulation.
   (c) Applications of the existence theorem, functional coercivity.
   (d) Weak lower semi-continuous functionals, problems in the vector case and their solution.
4. **Problems in weak formulation**
   (a) Problems in finite dimensional spaces. Solution of equation $f(x) = 0$ on the ball, Brouwer's theorem and equation $f(x) = y$ in $\mathbb{R}^N$.
   (b) Operator equation $A(u) = b$ in Banach space, convergence, continuity conditions, monotony and coercivity conditions.
   (c) Equation with strongly monotone operator, proof.
   (d) Galerkin approximation, properties of solution sequence.
   (e) Equation with weakly continuous operator, proof.
   (f) Stationary nonlinear diffusion.

IV. Introduction to stochastic differentia equations

1. **Basic notions**
   (a) Probability, random variable, expected value and variance, normal distribution, independence, random process.
   (b) Wiener process (Brown motion), properties, linear and quadratic variance.
2. **Stochastic integral and differential equations**
   (a) Introduction of Itô and Stratonovich integral.
   (b) Differential of composed functions, Itô formule.
   (c) Solution of Itô stochastic differential equation.

Exam

1. **Practical part** — given boundary value problem for a quasi-linear equation in classic, weak or variational formulation.
   (a) Derive the other formulations.
   (b) Complete the assumptions for data in order to the formulation is correct.
   (c) Add conditions ensuring existence and uniqueness of the solution.
2. **Theoretical part** — one question from parts I, II, III and IV.

Example: Let $\Omega = (-1,1)^2$. Let find minimum on the set $V = \{u \in W^{1,2}(\Omega) : u(\pm 1, y) = 1\}$ of the integral functional

$$\Phi(u) = \int_\Omega \left[u_x^2 + 2\,u_y^2 + a(x,y,u) + fu\right] \mathrm{d}x\,\mathrm{d}y + \int_{-1}^{1} (u^2 - gu)(x, \pm 1)\,\mathrm{d}x.$$

Derive weak and classical formulation of the problem. Complete the assumption for functions a, f, g, in order to the formulace be correct. When existence and uniqueness of the solution is ensured?

Recommended literature for students

1. J. Franců: *Moderní metody řešení diferenciálních rovnic*, skripta FSI VUT, Akad. nakl. CERM, Brno 2006.
2. E. Kolářová: *Stochastické diferenciální rovnice v elektrotechnice* (vybrané stránky z disertační práce na FSI VUT – dostupné na internetové adrese: `http://www.mat.fme.vutbr.cz/home/francu/`).

Brno, March 6^{th}, 2019 Jan Franců