Topics for Master's State Final Exam

Familiarity with the mathematics taught in Bachelor's degree courses is expected.

I Theory of graphs, logic and mathematical structures

1) **Graphs:** representation of graphs, subgraphs, vertex degree, paths and cycles, connectivity of graphs, isomorphism of graphs.
2) **Special graphs:** trees, planar graphs, chromatic number of a graph, Hamiltonian and Eulerian graphs, directed graphs, tournaments.
3) **Optimization problems on graphs:** algorithms for problems solvable in a polynomial time (minimal spanning tree, shortest paths in graphs, maximal flow in a network).
4) **Propositional calculus:** logical connectives and formulas, duality principle, applications (electrical circuits), complete systems and bases of connectives, axiomatization of propositional calculus, completeness theorem.
5) **Predicate calculus:** languages (terms, atomic formulas, and formulas) and semantics (language implementation and variable valuation), logically valid formulas.
6) **Axiomatic system of predicate calculus:** axioms and derivation rules, proving formulas, theorem on deduction, theorems on completeness and consistency.
7) **Mathematical structures**: constructs and subconstructs, isomorphisms, fibres, subobjects and generation, quotient objects, free objects, initial structures and Cartesian products, final structures.

II Function of complex variable, partial differential equations and mathematical methods in fluid dynamics

1) **Functions of complex variable:** limit, continuity, definition and properties of elementary functions in the complex domain.
2) **Derivatives of functions in the complex domain:** derivative and its geometric meaning, holomorphic functions, Taylor and Laurent series, singular points.
3) **Integral in complex domain:** antiderivative, independence of the integration path, Cauchy theorem, Cauchy integral formula, theorem on the uniqueness of holomorphic functions, residua and their use.
4) **Discrete Fourier analysis and its applications in image processing:** discrete Fourier transform (DFT) and its properties, convolution and the convolution theorem, frequency domain representation of images, filtering in the frequency domain (low-pass, high-pass and band-pass filters).
5) **Classical theory of partial differential equations:** linear and semi-linear first-order PDE (method of characteristics), classification of second-order equations, canonical forms, well posed problems, initial-boundary value problems.
6) **Solution of second-order partial differential equations:** heat equation (derivation, maximum principle, fundamental solution). Laplace and Poisson equations (maximum principle, fundamental solution and Green's function), wave equation (one dimensional equation, fundamental solution and D'Alembert formula, fundamental solution in three dimensions and strong Huygens' principle, Kirchoff formula).
7) **Equations of mathematical physics:** basics and constitutional relations, string vibration and wave equations, heat conduction and diffusion equations, equation of the deflection of a membrane, their derivation. Formulations of initial and boundary-value problems and their physical interpretations.

8) **Elements of the modern theory of partial differential equations:** distributions (generalized functions), Sobolev spaces, embedding theorems, weak and variational formulation of elliptic problems, Lax–Milgram lemma.
9) **Methods of nonlinear analysis:** topological methods (Brouwer's theorem, Schauder's theorem), theory of monotone operators ((strict/strong) monotonicity, (weak) coercivity), variational methods (critical points, weakly sequentially lower semi-continuous functionals, weakly sequentially compact sets), applications in differential equations.
10) **Mathematical Methods in Fluid Dynamics:** conservation laws (mass, momentum and energy), deriving the Navier-Stokes and Euler equations, hyperbolic systems, classical and weak solution, Finite Volume Method, stability and CFL condition, Godunov numerical flux.

III Optimization and variational calculus

1) **Finite-dimensional optimization of linear and nonlinear problems:** basic properties of convex sets and convex functions, formulations of LP and NLP problems, optimality criteria, key ideas of simplex method.
2) **Basic concepts of financial mathematics:** Key computational ideas of fundamental concepts in financial mathematics (simple and compound interest, savings and pensions). Principal ideas of Markowitz portfolio optimization.
3) **Optimal control of dynamical systems:** eligible and optimal regulations, basic problem of optimal control, maximum principle, transversality conditions, applications.
4) **Linear problem of time optimization:** necessary and sufficient optimality conditions, basic properties of optimal regulations and trajectories.

IV Functional analysis and numerical methods

1) **Measure and integral:** construction of the Lebesgue measure on $\mathbb{R}$ and of the Lebesgue integral, Lebesgue's dominated convergence theorem, monotone convergence theorem, Lebesgue spaces and their properties.
2) **Metric spaces:** metric, convergence, complete spaces, Banach fixed point theorem, compact spaces, examples of metric spaces.
3) **Normed linear spaces and inner product spaces:** norm, inner product, Banach space, Hilbert space, abstract Fourier series, Bessel inequality, Parseval equality, Riesz–Fischer theorem.
4) **(Continuous) linear functionals**: norm of a functional, Hahn–Banach theorem, dual spaces, representation of functionals, weak convergence, (in particular, spaces of sequences and functions).
5) **Linear operators**: compact operator, invertibility, adjoint operator, operator spectrum, Fredholm theory in Hilbert space
6) **Numerical methods:** solving a single non-linear equation, solving systems of non-linear solutions, solving systems of linear equations (direct and iteration method, LU, Cholesky, QR decompositions), interpolation (Lagrange and Hermit), numerical differentiation and integration, least-squares method.

V **Probability, random variable, stochastic processes, and dynamical systems**

1) **Probability:** random events, field of events, axiomatic probability, properties of probability, conditional probability, independence.
2) **Random variable:** definition, distribution function, discrete and continuous random variables, numerical characteristics of random variables and their properties, quantiles, basic probability distributions (Bi, H, Po, N, W), characteristic functions, law of large numbers, central limit theorem
3) **Stochastic processes: d**iscrete time processes (Markov chains in finite and countable space, limiting distribution), continuous time processes (density and distribution of into-event time for Poisson process, applications and extensions), Wiener processes and basic stochastic calculus (basic definitions and properties, Itô's formula, stochastic differential equations).
4) **Nonlinear ODEs and systems of ODEs:** first-order nonlinear ODE, first-order linear systems of autonomous ODE, equilibria, existence of solutions, linear stability.
5) **Dynamical systems:** stability, the first (indirect) method, the second (direct/Lyapunov function) method, planar non-hyperbolic equilibrium points, Poincaré–Bendixson theorem, normal form theory, limit sets and attractors, Hamiltonian systems, bifurcation (bifurcation at non-hyperbolic equilibrium points, Hopf bifurcation), structural stability.

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