Topics for Master's State Final Exam

Familiarity with the mathematics taught in Bachelor's degree courses is expected.

I Theory of graphs, logic and mathematical structures

1) **Graphs:** representation of graphs, subgraphs, vertex degree, paths and cycles, connectivity of graphs, isomorphism of graphs.
2) **Special graphs:** trees, planar graphs, chromatic number of a graph, Hamiltonian and Eulerian graphs, directed graphs, tournaments.
3) **Optimization problems on graphs:** algorithms for problems solvable in a polynomial time (minimal spanning tree, shortest paths in graphs, maximal flow in a network).
4) **Propositional calculus:** logical connectives and formulas, duality principle, applications (electrical circuits), complete systems and bases of connectives, axiomatization of propositional calculus, completeness theorem.
5) **Predicate calculus:** languages (terms, atomic formulas, and formulas) and semantics (language implementation and variable valuation), logically valid formulas.
6) **Axiomatic system of predicate calculus:** axioms and derivation rules, proving formulas, theorem on deduction, theorems on completeness and consistency.
7) **Mathematical structures**: constructs and subconstructs, isomorphisms, fibres, sub-objects and generation, quotient objects, free objects, initial structures and Cartesian products, final structures.

II Functions of complex variable, partial differential equations and mathematical methods in fluid dynamics

1) **Functions of complex variable:** limit, continuity, definition and properties of elementary functions in the complex domain.
2) **Derivatives of functions in the complex domain:** derivative and its geometric meaning, holomorphic functions, Taylor and Laurent series, singular points.
3) **Integral in complex domain:** antiderivative, independence of the integration path, Cauchy theorem, Cauchy integral formula, theorem on the uniqueness of holomorphic functions, residua and their use.
4) **Classical theory of partial differential equations:** linear and semilinear first-order PDE (method of characteristics), classification of second-order equations, canonical forms, well posed problems, initial-boundary value problems.
5) **Solution of second-order partial differential equations:** heat equation (derivation, maximum principle, fundamental solution). Laplace and Poisson equations (maximum principle, fundamental solution and Green function), wave equation (one dimensional equation, fundamental solution and D'Alembert formula, fundamental solution in three dimensions and strong Huygens' principle, Kirchoff formula).
6) **Equations of mathematical physics:** basics and constitutional relations, string vibration and wave equations, heat conduction and diffusion equations, equation of the deflection of a membrane, their derivation. Formulations of initial and boundary-value problems and their physical interpretations.
7) **Modern solution methods for partial differential equations:** spaces of functions, weak and variational formulation of elliptic problems, Lax–Milgram lemma, specific features of nonlinear problems.

8) **Mathematical Methods in Fluid Dynamics:** conservation law of mass, momentum and energy, Navier-Stokes and Euler equations, hyperbolic system, classical and weak solution, Finite volume method, stability and CFL condition, Godunov numerical flux.

III <u>**Optimization and variational calculus**</u>

1) **Finite-dimensional optimization of linear problems – basics:** convex and polyhedral sets, formulation of LP problems.
2) **Finite-dimensional optimization of linear problems – solution methods:** representation of the set of feasible solutions, criteria for optimality, simplex method.
3) **Finite-dimensional optimization of nonlinear problems – basics:** convex functions and their properties, formulation of NLP problems.
4) **Finite-dimensional optimization of nonlinear problems – solution methods:** Karush–Kuhn–Tucker conditions of the existence of extremes and their geometric interpretation, basic numerical algorithms.
5) **Optimal control of dynamical systems:** eligible and optimal regulations, basic problem of optimal control, maximum principle, transversality conditions, applications.
6) **Linear problem of time optimization:** necessary and sufficient optimality conditions, basic properties of optimal regulations and trajectories.

IV <u>**Functional analysis and numerical methods**</u>

1) **Measure and integral:** construction of the Lebesgue measure on R and of the Lebesgue integral, Lebesgue's dominanted convergence theorem, monotone convergence theorem, Lebesgue spaces and their properties.
2) **Metric spaces:** metric, convergence, complete spaces, Banach fixed point theorem and its applications, compact spaces, examples (in particular, spaces of sequences and functions).
3) **Normed linear spaces and inner product spaces:** norm, inner product, Banach space, Hilbert space, (in particular, spaces of sequences and functions), Hamel basis vs. Schauder basis, abstract Fourier series, Riesz–Fischer theorem.
4) **(Continuous) linear functionals**: norm of a functional, Hahn–Banach theorem, dual spaces, representation of functionals, weak convergence, (in particular, spaces of sequences and functions).
5) **Numerical methods for solving algebraic equations:** solving a single non-linear equation, solving systems of non-linear solutions, solving systems of linear equations (direct and iteration method).
6) **Interpolation and approximation, numerical differentiation and integration:** notions of interpolation and approximation, Lagrange and Hermit interpolation polynomials in one and several variables, least-squares method.

V <u>**Probability and dynamical systems**</u>

1) **Probability:** random events, field of events, axiomatic probability, properties of probability, conditional probability, independence.
2) **Random variable:** definition, distribution function, discrete and continuous random variables, numerical characteristics of random variables and their properties,

quantiles, basic probability distributions (Bi, H, Po, N, W), characteristic functions, law of large numbers, central limit theorem.

3) **Random vector:** definition, simultaneous distribution function, discrete and continuous random vector, marginal and conditional distribution, independence, numerical characteristics, transformations of random vectors, multivariate normal distribution, characteristic function.
4) **Nonlinear ODEs and systems of ODEs:** first-order nonlinear ODE, first-order linear systems of autonomous ODE, equilibria, existence of solutions.
5) **Local and global theory:** linearization, stability and Lyapunov functions, stable manifold theorem, planar non-hyperbolic critical points, center manifold theory, normal form theory, limit sets and attractors, limit cycles and separatrix cycles, Poincaré map, Hamiltonian systems, Poincaré–Bendixson theorem.
6) **Bifurcation theory**: structural stability, bifurcation at non-hyperbolic equilibrium points, Hopf bifurcation.

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