Numerická derivace

1) $f(x)$ známe jen v tabulkových bodech a vstupní data nezatížena chybou
2) $f(x)$ známe jen v tabulkových bodech a vstupní data zatížena chybou
3) $f(x)$ je známa, ale na přímou derivaci příliš složitá

Princip: náhrada neznámé funkce $f(x)$ její aproximací $g(x)$ a derivace této aproximace

ad 1: Příklad: Pomocí Lagrangeova interpolačního polynomu určete hodnotu $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$ v bodě $x_0 = 5$

Řešení: $y = \sqrt{x}$

x_i	1	4	9
y_i	1	2	3

$$\sqrt{x} \approx L_2(x) = \frac{1}{24}\cdot(x-4)(x-9) - \frac{2}{15}\cdot(x-1)(x-9) + \frac{3}{40}\cdot(x-1)(x-4)$$
$$= \frac{1}{24}\cdot(x^2-13x+36) - \frac{2}{15}\cdot(x^2-10x+9) + \frac{3}{40}\cdot(x^2-5x+4)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} \approx \frac{\mathrm{d}}{\mathrm{d}x}L_2(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{24}\cdot(x^2-13x+36) - \frac{2}{15}\cdot(x^2-10x+9) + \frac{3}{40}\cdot(x^2-5x+4)\right] =$$
$$= \frac{1}{24}\cdot(2x-13) - \frac{2}{15}\cdot(2x-10) + \frac{3}{40}\cdot(2x-5)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}L_2(5) = \frac{1}{24}\cdot(2\cdot5-13) - \frac{2}{15}\cdot(2\cdot5-10) + \frac{3}{40}\cdot(2\cdot5-5) = -\frac{3}{24} + \frac{15}{40} = -\frac{1}{8} + \frac{3}{8} = 0.25$$
$$f'(x_0) = \frac{1}{2\sqrt{x_0}} = \frac{1}{2\sqrt{5}} \approx 0.224$$

ad 2: Příklad: Při měření dráhy pohybujícího se tělesa byly v časových okamžicích t_i zjištěny následující hodnoty. Určete okamžitou rychlost tělesa v čase $t = 3.5$!

i	0	1	2	3	4	5
t_i	0	1	2	3	4	5
s_i	5.2	10.1	24.9	50.0	85.2	130.0
$\varphi_0(t_i)$	1	1	1	1	1	1
$\varphi_1(t_i)$	0	1	2	3	4	5
$\varphi_2(t_i)$	0	1	4	9	16	25

Měřené hodnoty $\Rightarrow$ MNČ

$$s(t) = a_0 \quad 1 \quad + a_1 \quad t \quad + a_2 \quad t^2$$
$$s(t) = a_0\varphi_0(t) + a_1\varphi_1(t) + a_2\varphi_2(t)$$

$$\begin{pmatrix} (\boldsymbol{\varphi_0};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_0};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_1};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_1}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) \\ (\boldsymbol{\varphi_2};\boldsymbol{\varphi_0}) & (\boldsymbol{\varphi_1};\boldsymbol{\varphi_2}) & (\boldsymbol{\varphi_2};\boldsymbol{\varphi_2}) \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\varphi_0};\boldsymbol{s}) \\ (\boldsymbol{\varphi_1};\boldsymbol{s}) \\ (\boldsymbol{\varphi_2};\boldsymbol{s}) \end{pmatrix}$$

ad 2: Příklad: Při měření dráhy pohybujícího se tělesa byly v časových okamžicích t_i zjištěny následující hodnoty. Určete okamžitou rychlost tělesa v čase $t = 3.5$!

i	0	1	2	3	4	5
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Měřené hodnoty $\Rightarrow$ MNČ

$$s(t) = a_0 \quad 1 \quad + a_1 \quad t \quad + a_2 \quad t^2$$
$$s(t) = a_0\varphi_0(t) + a_1\varphi_1(t) + a_2\varphi_2(t)$$

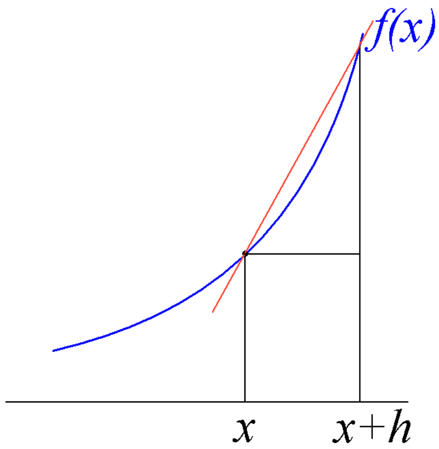
$$\begin{pmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 305.4 \\ 1200.7 \\ 5172.9 \end{pmatrix}$$

$a_0 = 5.02$; $\quad a_1 = -0.115$; $a_2 = 5.09$

$\boxed{s(t) = 5.0 - 0.1 \cdot t + 5.1 \cdot t^2}$ $\quad v(t) = \dot{s}(t) = -0.1 + 2 \cdot 5.1 \cdot t \quad v(3.5) = -0.1 + 2 \cdot 5.1 \cdot 3.5 = 35.6$

Možnosti pro derivaci v bodě:

dopředná diference $\qquad f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0)}{h}$



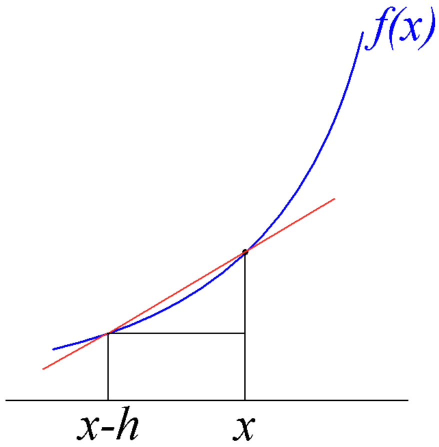
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{3x}\sin\sqrt{5x} \quad \text{v bodě } x_0 = 5$$

$x_0 = 5;\ h = 0.05$

$$f'(x_0) \approx \frac{\sqrt{3(x_0+h)} \cdot \sin\sqrt{5(x_0+h)} - \sqrt{3x_0} \cdot \sin\sqrt{5x_0}}{h} = \frac{\sqrt{3 \cdot 5.05} \cdot \sin\sqrt{5 \cdot 5.05} - \sqrt{3 \cdot 5} \cdot \sin\sqrt{5 \cdot 5}}{0.05} \approx 0.2034$$

Možnosti pro derivaci v bodě:

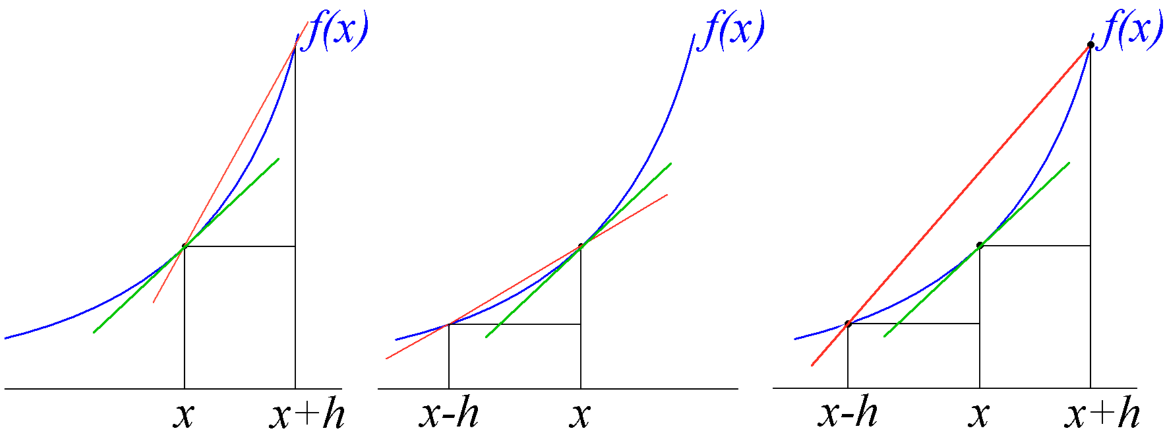
zpětná diference $\quad f'(x_0) \approx \dfrac{f(x_0) - f(x_0 - h)}{h}$



$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{3x} \sin \sqrt{5x} \quad \text{v bodě } x_0 = 5$$

$x_0 = 5;\ h = 0.05$

$$f'(x_0) \approx \frac{\sqrt{3x_0} \cdot \sin\sqrt{5x_0} - \sqrt{3(x_0 - h)} \cdot \sin\sqrt{5(x_0 - h)}}{h} = \frac{\sqrt{3 \cdot 5} \cdot \sin\sqrt{5 \cdot 5} - \sqrt{3 \cdot 4.95} \cdot \sin\sqrt{5 \cdot 4.95}}{0.05} \approx 0.1524$$



$f'(x_0) \approx \dfrac{f(x_0 + h) - f(x_0)}{h}$ $\qquad f'(x_0) \approx \dfrac{f(x_0) - f(x_0 - h)}{h}$ $\qquad f'(x_0) \approx \dfrac{f(x_0 + h) - f(x_0 - h)}{2h}$

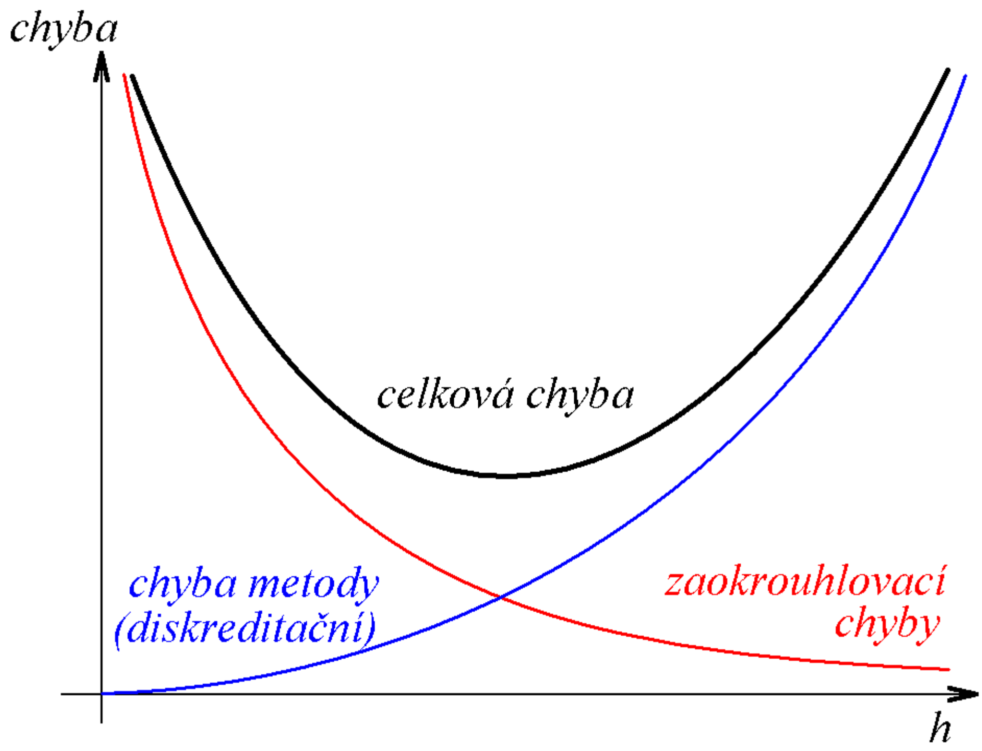
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sqrt{3x} \sin \sqrt{5x} \quad \text{v bodě } x_0 = 5$$

$x_0 = 5;\ h = 0.05$

$$f'(x_0) \approx \frac{\sqrt{3 \cdot (x_0 + h)} \cdot \sin\sqrt{5(x_0 + h)} - \sqrt{3(x_0 - h)} \cdot \sin\sqrt{5(x_0 - h)}}{2h} =$$

$$= \frac{\sqrt{3 \cdot 5.05} \cdot \sin\sqrt{5 \cdot 5.05} - \sqrt{3 \cdot 4.95} \cdot \sin\sqrt{5 \cdot 4.95}}{2 \cdot 0.05} \approx 0.1779$$

Chyby numerické derivace:



ad 3: Richardsonova metoda:

centrální $f'(x_0) \approx \dfrac{f(x_0+h)-f(x_0-h)}{2h}$ $\boxed{g_1(h) = \dfrac{f(x_0+h)-f(x_0-h)}{2h}}$

$$f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0} g(h)$$

$$-\left[g(h) = a_0 + a_1 h^2 + a_2 h^4 + ...\right] \qquad h \to 0 \Rightarrow g(h) \to a_0$$

$h \to 0$

$$\left[g\left(\frac{h}{2}\right) = a_0 + \frac{1}{4} a_1 h^2 + \frac{1}{16} a_2 h^4 + ...\right] \cdot 4$$

$$4 \cdot g_1\left(\frac{h}{2}\right) - g_1(h) = 3 \cdot a_0 \qquad \underbrace{-\frac{3}{4} a_2 h^4 + ...}_{\to 0} \Rightarrow a_0 \approx \frac{1}{3} \cdot \left[4 \cdot g_1\left(\frac{h}{2}\right) - g_1(h)\right]$$

$$\boxed{g_2(h) = \frac{1}{3} \cdot \left[4 \cdot g_1\left(\frac{h}{2}\right) - g_1(h)\right]}$$

$$-\left[g_2(h) = a_0 + \bar{a}_1 h^4 + \bar{a}_2 h^6 + ...\right]$$

$$\left[g_2\left(\frac{h}{2}\right) = a_0 + \bar{a}_1 \frac{h^4}{16} + \bar{a}_2 \frac{h^6}{64} + ...\right] \cdot 16$$

$$16 \cdot g_2\left(\frac{h}{2}\right) - g_2(h) = 15 \cdot a_0 + \underbrace{\bar{a}_2 \frac{h^6}{4} ......}_{\to 0} \Rightarrow a_0 \approx \frac{1}{15} \cdot \left[16 \cdot g_2\left(\frac{h}{2}\right) - g_2(h)\right]$$

$$\boxed{g_3(h) = \frac{1}{15} \cdot \left[16 \cdot g_2\left(\frac{h}{2}\right) - g_2(h)\right]}$$

ad 3: Richardsonova metoda:

$$g_1(h) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$g_2(h) = \frac{1}{3}\cdot\left[4\cdot g_1\left(\frac{h}{2}\right) - g_1(h)\right]$$

$$g_3(h) = \frac{1}{15}\cdot\left[16\cdot g_2\left(\frac{h}{2}\right) - g_2(h)\right]$$

$$g_{k+1}(h) = \frac{1}{4^k - 1}\cdot\left[4^k\cdot g_k\left(\frac{h}{2}\right) - g_k(h)\right]$$

Schématický zápis:

$g_1(h)$
$g_1\left(\tfrac{h}{2}\right)\ \ g_2(h)$
$g_1\left(\tfrac{h}{4}\right)\ \ g_2\left(\tfrac{h}{2}\right)\ \ g_3(h)$
$g_1\left(\tfrac{h}{8}\right)\ \ g_2\left(\tfrac{h}{4}\right)\ \ g_3\left(\tfrac{h}{4}\right)\ \ g_4(h)$

ad 3: Richardsonova metoda:

$$g_1(h) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$g_2(h) = \frac{1}{3}\cdot\left[4\cdot g_1\left(\frac{h}{2}\right) - g_1(h)\right]$$

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Schématický zápis:

$g_1(h)$ 0,177 850 965
$g_1\left(\tfrac{h}{2}\right)\ \ g_2(h)$ 0,177 902 504 0,177 919 684
$g_1\left(\tfrac{h}{4}\right)\ \ g_2\left(\tfrac{h}{2}\right)\ \ g_3(h)$
$g_1\left(\tfrac{h}{8}\right)\ \ g_2\left(\tfrac{h}{4}\right)\ \ g_3\left(\tfrac{h}{4}\right)\ \ g_4(h)$

$$f(x) = \sqrt{3x}\sin\sqrt{5x};\quad x_0 = 5;\quad h = 0{,}05$$

$$f'(5) = \frac{3\sin\sqrt{5\cdot 5}}{2\sqrt{3\cdot 5}} + \frac{5\sqrt{3\cdot 5}\cos\sqrt{5\cdot 5}}{2\sqrt{5\cdot 5}} \approx 0.177\ 919\ 686$$

$$g_2(h) = \frac{1}{3}\cdot\left[4\cdot g_1\left(\frac{h}{2}\right) - g_1(h)\right] = \frac{1}{3}\cdot\left[4\cdot 0{,}177\ 902\ 504 - 0{,}177\ 850\ 965\right] = 0{,}177\ 919\ 684$$

Numerická integerace

Důvody pro numerickou derivaci

1) $f(x)$ známe jen v tabulkových bodech a vstupní data nezatížena chybou
2) $f(x)$ známe jen v tabulkových bodech a vstupní data zatížena chybou
3) $f(x)$ je známa, ale na přímou derivaci příliš složitá

Důvody pro numerickou integeraci

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2) $f(x)$ známe jen v tabulkových bodech a vstupní data zatížena chybou
3) $f(x)$ je známa, ale na přímou integraci příliš složitá
4) **$f(x)$ nelze přímo počítat**

$$\int_a^b \frac{\sin x}{x} dx = ????????$$

Obecný postup:

Integrand $f(x)$ nahradíme aproximací $\varphi(x)$ a hodnotu $\int_a^b f(x)dx$ nahradíme hodnotou $\int_a^b \varphi(x)dx$

$$\int_a^b \varphi(x)dx \approx \int_a^b f(x)dx$$

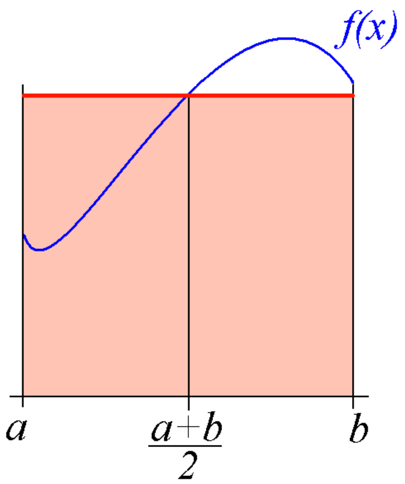
$$Q(f) \approx \int_a^b f(x)dx \qquad Q(f) \text{ - } \textbf{kvadraturní formule.}$$

$$Q(f) + R(f) = \int_a^b f(x)dx \Rightarrow R(f) = \boxed{\int_a^b f(x)dx - Q(f)} \qquad \textbf{diskretizační chyba}$$

Základní kvadraturní formule

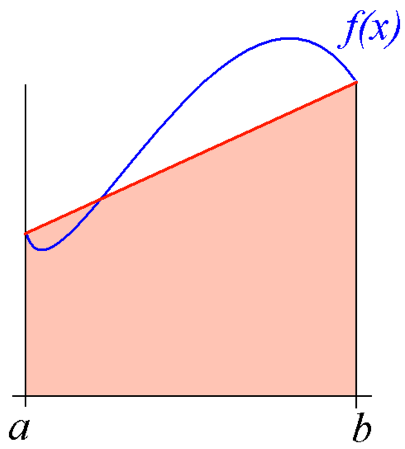
(integrace interpolačního polynomu) $\int_a^b f(x)dx \approx \int_a^b L_n(x)dx$

$n=0$ - obdélníková formule



$$\int_a^b f(x)dx \approx (b-a)\cdot f\left(\frac{a+b}{2}\right)$$

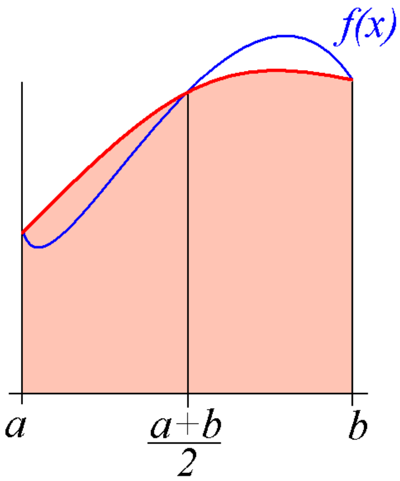
$n=1$ - lichoběžníková formule

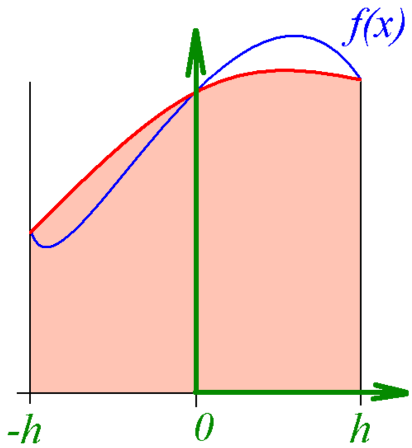


$$\int_a^b f(x)dx \approx \frac{b-a}{2}\cdot\left[f(a)+f(b)\right]$$

Chyba: $E_M \le \frac{1}{24}\max_{x\in\langle a;b\rangle}|f''(x)|(b-a)^3$; $E_L \le \frac{1}{12}\max_{x\in\langle a;b\rangle}|f''(x)|(b-a)^3$

$n=2$ - Simpsonova formule





$$y=Ax^2+Bx+C \Rightarrow \int_a^b f(x)dx \approx \int_{-h}^{h} y dx = \int_{-h}^{h}\left(Ax^2+Bx+C\right)dx = \left[\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right]_{-h}^{h} =$$

$$=\frac{Ah^3}{3}+\frac{Bh^2}{2}+Ch-\left(\frac{A(-h)^3}{3}+\frac{B(-h)^2}{2}+C(-h)\right)=\boxed{\frac{2}{3}Ah^3+2Ch} \quad \Rightarrow \quad \int_a^b f(x)dx \approx \boxed{\frac{h}{3}\left(2Ah^2+6C\right)}$$

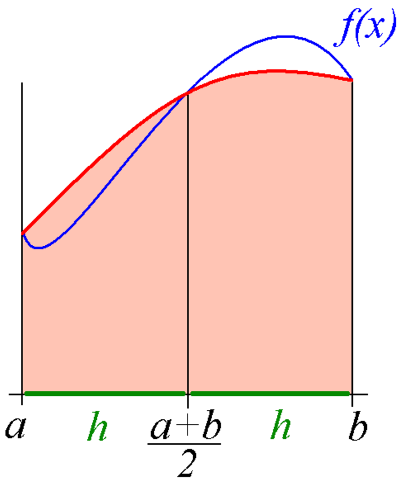
$$\begin{array}{llll} [-h; f(-h)]: & f(-h)= & Ah^2 \quad -Bh \quad +C & /:3 \\ [0; f(0)]: & f(0)= & C & /\cdot \tfrac{4}{3} \\ [h; f(h)]: & f(h)= & Ah^2 \quad +Bh \quad +C & /:3 \\ \hline \end{array}$$

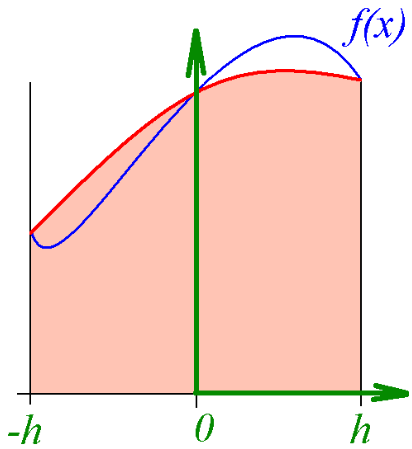
$$f(-h)+4f(0)+f(h)=2Ah^2+6C$$

$$f(-h)+4f(0)+f(h)=2Ah^2+6C$$

$$\boxed{\int_a^b f(x)dx \approx \frac{h}{3}\left(f(-h)+4f(0)+f(h)\right)}$$

$n=2$ - Simpsonova formule





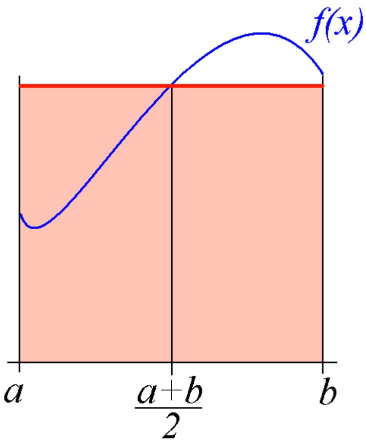
$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

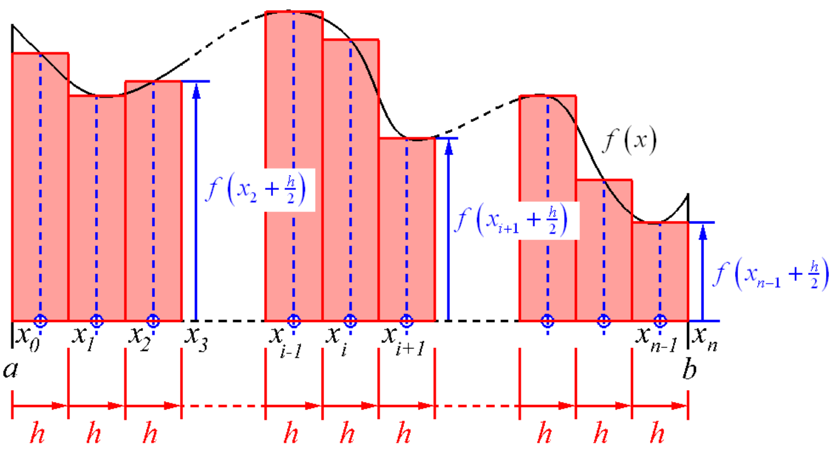
$$\boxed{\int_a^b f(x)dx \approx \frac{h}{3}\left(f(-h)+4f(0)+f(h)\right)}$$

Chyba: $E_S \le \frac{1}{90}\max_{x\in\langle a;b\rangle}\left|f^{(4)}(x)\right|\cdot\left(\frac{b-a}{2}\right)^5$

Složené formule

Obdélníková:





$$\int_a^b f(x)\mathrm{d}x = \int_{x_0}^{x_1} f(x)\mathrm{d}x + \int_{x_1}^{x_2} f(x)\mathrm{d}x + \cdots + \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x \approx h\cdot f\left(x_0+\frac{h}{2}\right) + h\cdot f\left(x_1+\frac{h}{2}\right) + \cdots + h\cdot f\left(x_{i-1}+\frac{h}{2}\right) + h\cdot f\left(x_i+\frac{h}{2}\right) + \cdots + h\cdot f\left(x_{n-1}+\frac{h}{2}\right)$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx h\cdot \left[f\left(x_0+\frac{h}{2}\right) + f\left(x_1+\frac{h}{2}\right) + \cdots + f\left(x_{i-1}+\frac{h}{2}\right) + f\left(x_i+\frac{h}{2}\right) + \cdots + f\left(x_{n-1}+\frac{h}{2}\right)\right]}$$

Složené formule

Obdélníková:

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx h\cdot \left[f\left(x_0+\frac{h}{2}\right) + f\left(x_1+\frac{h}{2}\right) + \cdots + f\left(x_{i-1}+\frac{h}{2}\right) + f\left(x_i+\frac{h}{2}\right) + \cdots + f\left(x_{n-1}+\frac{h}{2}\right)\right]}$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}\cdot \left[f\left(x_0+\frac{h}{2}\right) + f\left(x_1+\frac{h}{2}\right) + \cdots + f\left(x_{i-1}+\frac{h}{2}\right) + f\left(x_i+\frac{h}{2}\right) + \cdots + f\left(x_{n-1}+\frac{h}{2}\right)\right]}$$

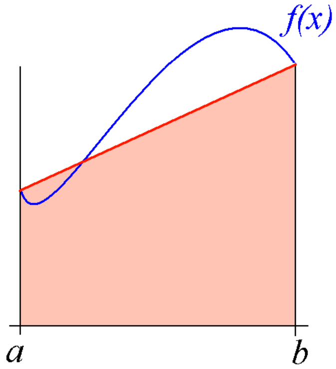
Chyba: **prostá** **složená:**

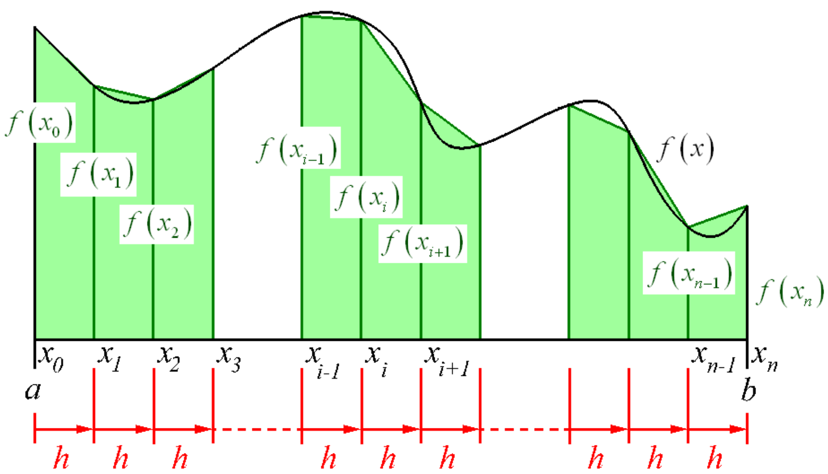
$$E_O \le \frac{1}{24}\max_{x\in\langle a;b\rangle}|f''(x)|\cdot(b-a)^3 \qquad E_O \le \frac{1}{24}\max_{x\in\langle a;b\rangle}|f''(x)|\cdot(b-a)\cdot h^2$$

Ukázka

Složené formule

Lichoběžníková





$$\int_a^b f(x)\mathrm{d}x = \int_{x_0}^{x_1} f(x)\mathrm{d}x + \int_{x_1}^{x_2} f(x)\mathrm{d}x + \dots + \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x + \dots + \int_{x_{n-1}}^{x_n} f(x)\mathrm{d}x$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\cdot[f(x_0)+f(x_1)] + \frac{h}{2}\cdot[f(x_1)+f(x_2)] + \dots + \frac{h}{2}\cdot[f(x_{i-1})+f(x_i)] + \dots + \frac{h}{2}\cdot[f(x_{n-1})+f(x_n)]$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\cdot[\, f(x_0) + 2\cdot f(x_1) + \dots + 2\cdot f(x_i) + \dots + 2\cdot f(x_{n-1}) + f(x_n)\,]$$

Složené formule

Lichoběžníková

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}\cdot[\, f(x_0) + 2\cdot f(x_1) + \dots + 2\cdot f(x_i) + \dots + 2\cdot f(x_{n-1}) + f(x_n)\,]}$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx h\cdot\left[\frac{f(x_0)}{2} + f(x_1) + \dots + f(x_i) + \dots + f(x_{n-1}) + \frac{f(x_n)}{2}\right]}$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{2n}\cdot[\, f(x_0) + 2\cdot f(x_1) + \dots + 2\cdot f(x_i) + \dots + 2\cdot f(x_{n-1}) + f(x_n)\,]}$$

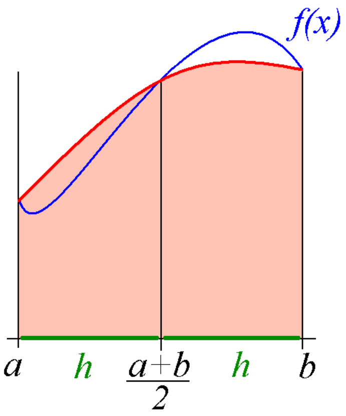
$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{n}\cdot\left[\frac{f(x_0)}{2} + f(x_1) + \dots + f(x_i) + \dots + f(x_{n-1}) + \frac{f(x_n)}{2}\right]}$$

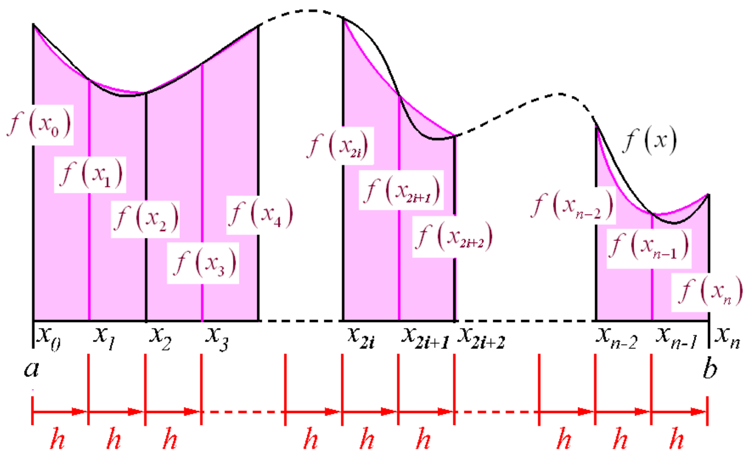
Chyba: **prostá:** **složená:**

$$E_L \le \frac{1}{12}\max_{x\in\langle a;b\rangle}|f''(x)|\cdot(b-a)^3 \qquad E_L \le \frac{1}{12}\max_{x\in\langle a;b\rangle}|f''(x)|\cdot(b-a)\cdot h^2$$

Složené formule

Simpsonova





$$\int_a^b f(x)\mathrm{d}x = \int_{x_0}^{x_2} f(x)\mathrm{d}x + \int_{x_2}^{x_4} f(x)\mathrm{d}x + \cdots + \int_{x_{2i}}^{x_{2i+2}} f(x)\mathrm{d}x + \cdots + \int_{x_{n-2}}^{x_n} f(x)\mathrm{d}x$$

$$\approx \frac{h}{3}\cdot[f(x_0)+4f(x_1)+f(x_2)] + \frac{h}{3}\cdot[f(x_2)+4f(x_3)+f(x_4)] + \cdots + \frac{h}{3}\cdot[f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\cdot[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]}$$

Složené formule

Simpsonova

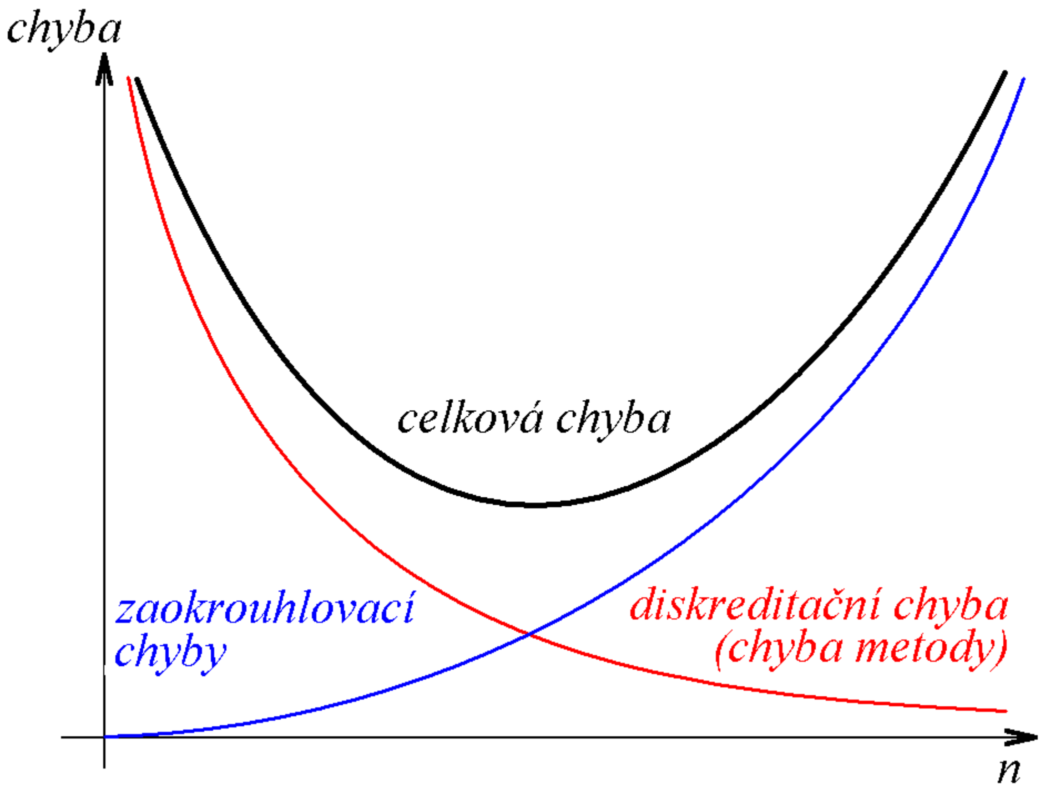
$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\cdot[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]}$$

$$\boxed{\int_a^b f(x)\mathrm{d}x \approx \frac{b-a}{3n}\cdot[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]}$$

Chyba: **prostá** **složená:**

$$E_S \le \frac{1}{90}\max_{x\in\langle a;b\rangle}\left|f^{(4)}(x)\right|\cdot\left(\frac{b-a}{2}\right)^5 \qquad E_S \le \frac{1}{90}\max_{x\in\langle a;b\rangle}\left|f^{(4)}(x)\right|\cdot\left(\frac{b-a}{2}\right)\cdot h^4$$

Celková chyba složených formulí:



Platí:

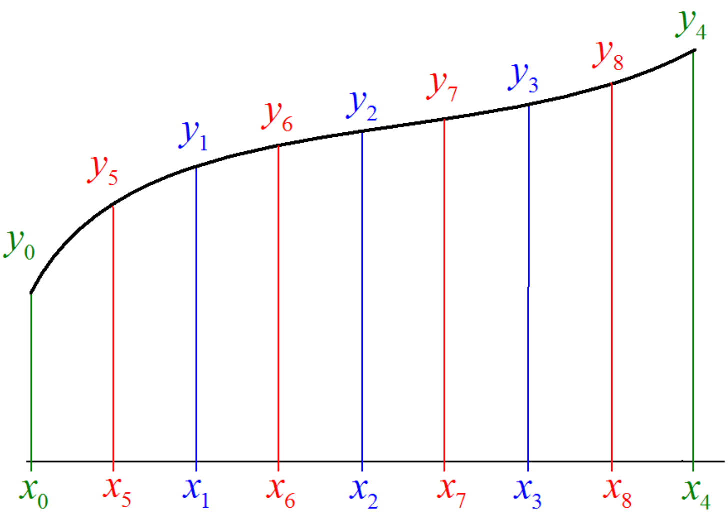
Kvadraturní formule získaná integrací Lagrangeova interpolačního polynomu stupně n má stupeň přesnosti **alespoň** n.

Řád kvadraturní formule je nanejvýš $2n+1$

Např:

Obdélníková formule : stupeň 0, přesnost 1
Lichoběžníková formule : stupeň 1, přesnost 1
Simpsonova formule : stupeň 2, přesnost 2

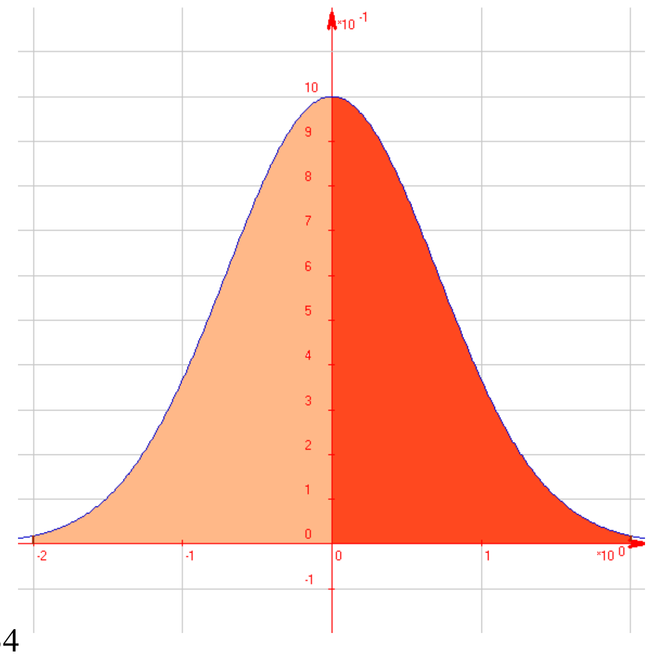
Metoda polovičního kroku:



$$\int_a^b f(x)\,\mathrm{d}x \approx h\cdot\left[\frac{1}{2}(y_0+y_n)+\sum_{i=1}^{n-1} y_i\right]$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{2}\cdot\left[\frac{1}{2}(y_0+y_n)+\sum_{i=1}^{n-1} y_i+\sum_{i=n+1}^{2n} y_i\right]$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{4}\cdot\left[\frac{1}{2}(y_0+y_n)+\sum_{i=1}^{n-1} y_i+\sum_{i=n+1}^{2n} y_i+\sum_{i=2n+1}^{4n} y_i\right]$$



Metoda polovičního kroku:

Hezký příklad:

$$\int_0^2 e^{-x^2}\,\mathrm{d}x = 0{,}88208139$$

$$\approx h\cdot\left[\tfrac{1}{2}(y_0+y_4)+(y_1+y_2+y_3)\right]=$$

$$=0.5\cdot\left[\tfrac{1}{2}\cdot\left(e^{-0^2}+e^{-2^2}\right)+\left(e^{-0.5^2}+e^{-1^2}+e^{-1.5^2}\right)\right]=$$

$$=0.5\cdot\left[\tfrac{1}{2}\cdot 1{,}018315639+1{,}252079449\right]=0{,}880618634$$

$$=0.25\cdot\left[\tfrac{1}{2}\left(e^{-0^2}+e^{-2^2}\right)+\left(e^{-0.5^2}+e^{-1^2}+e^{-1.5^2}\right)+\left(e^{-0.25^2}+e^{-0.75^2}+e^{-1.25^2}+e^{-1.75^2}\right)\right]=$$

$$=0.25\cdot\left[\tfrac{1}{2}1{,}018315639+1{,}252079449+1{,}765577897\right]=0{,}881703791$$

Odhad chyby:

$$\boxed{E_{2n}(f)=\frac{1}{2^{p+1}-1}\left|Q_{2n}(f)-Q_n(f)\right|}\quad E_{2n}(f)=\frac{1}{2^{1+1}-1}\left|0{,}881703791-0{,}880618634\right|\approx 0{,}00036$$

Metoda polovičního kroku - záludný příklad:

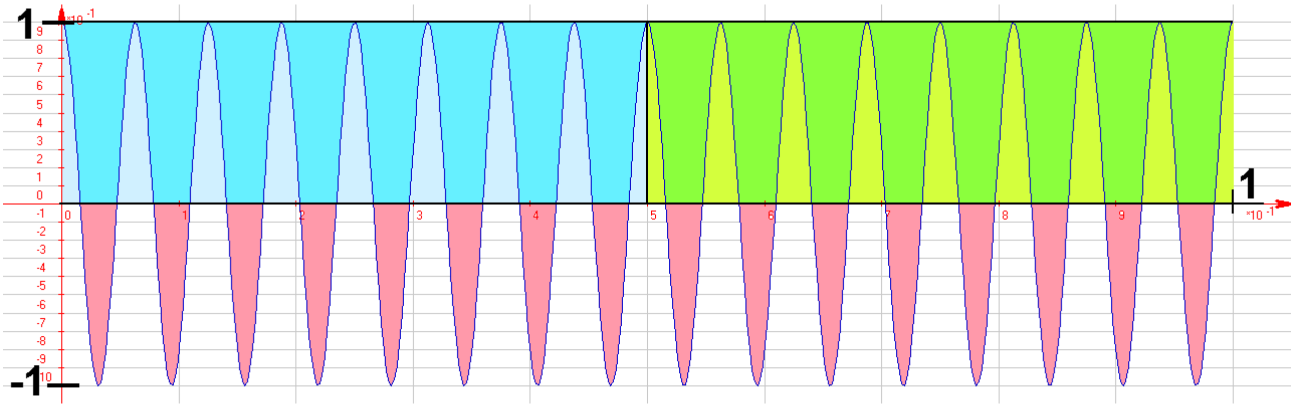
$$\int_0^1 \cos(64\pi x)\,\mathrm{d}x = \frac{1}{64\pi}\left[\sin(64\pi x)\right]_0^1 = \frac{1}{64\pi}\left[\sin(64\pi) - \sin 0\right] = 0$$

$$\approx h \cdot \left[y_0 + y_4 + \tfrac{1}{2}(y_1 + y_2 + y_3) \right] =$$

$$= \frac{1}{2} \cdot \left[\frac{1}{2}\cos(64\pi \cdot 0) + \frac{1}{2}\cos(64\pi \cdot 1) + \cos\left(64\pi \cdot \tfrac{1}{2}\right) \right] = 1$$

$$= \frac{1}{4} \cdot \left[\frac{1}{2}\cos(64\pi \cdot 0) + \frac{1}{2}\cos(64\pi \cdot 1) + \cos\left(64\pi \cdot \tfrac{1}{4}\right) + \cos\left(64\pi \cdot \tfrac{1}{2}\right) + \cos\left(64\pi \cdot \tfrac{3}{4}\right) \right] = 1$$

$$= \frac{1}{8} \cdot \Big[\frac{1}{2}\cos(64\pi \cdot 0) + \frac{1}{2}\cos(64\pi \cdot 1) + \cos\left(64\pi \cdot \tfrac{1}{4}\right) + \cos\left(64\pi \cdot \tfrac{1}{2}\right) + \cos\left(64\pi \cdot \tfrac{3}{4}\right) +$$
$$\cos\left(64\pi \cdot \tfrac{1}{8}\right) + \cos\left(64\pi \cdot \tfrac{3}{8}\right) + \cos\left(64\pi \cdot \tfrac{5}{8}\right) + \cos\left(64\pi \cdot \tfrac{7}{8}\right) \Big] = 1$$



Rombergova integrace:

$$\begin{array}{llll|l} T_{00} & & & & \\ T_{10} & T_{11} & & & \\ T_{20} & T_{21} & T_{22} & & \\ \hline T_{m0} & T_{m1} & T_{m2} & & T_{mm} \end{array}$$

T_{j0} - přibližné hodnoty integrálu získané lichoběžníkovou metodou pro 2^j dělicích bodů

$$T_{2,2} = \frac{4^2 T_{2,1} - T_{1,1}}{4^2 - 1} \qquad \text{obecně} \quad T_{j,k} = \frac{4^k T_{j,k-1} - T_{j-1,k-1}}{4^k - 1}$$

Příklad: $\int_0^\pi \sin x\,\mathrm{d}x$

$$T_{0;0}: \ 2^0 = 1; \quad n = 1; \quad \int_0^\pi \sin x\,\mathrm{d}x \approx \frac{\pi}{2}(\sin 0 + \sin \pi) = 0$$

$$T_{1;0}: \ 2^1 = 2; \quad n = 2; \quad \int_0^\pi \sin x\,\mathrm{d}x \approx \frac{\pi}{4}\left(\sin 0 + 2 \cdot \sin\frac{\pi}{2} + \sin \pi\right) = \frac{\pi}{2} \approx 1{,}570\ 796\ 327$$

$$T_{2;0}: \ 2^2 = 4; \quad n = 4; \quad \int_0^\pi \sin x\,\mathrm{d}x \approx \frac{\pi}{8}\left(\sin 0 + 2 \cdot \sin\frac{\pi}{4} + 2 \cdot \sin\frac{\pi}{2} + 2 \cdot \sin\frac{3\pi}{4} + \sin \pi\right) \approx 1{,}896\ 118\ 898$$

$$T_{1,1} = \frac{4^1 T_{1,0} - T_{0,0}}{4^1 - 1} = \frac{4 \cdot 1{,}570\ 796\ 327 - 0}{4 - 1} \approx 2{,}094\ 395\ 102$$

$$T_{2,1} = \frac{4^2 T_{2,0} - T_{1,0}}{4^2 - 1} = \frac{4^2 \cdot 1{,}896\ 118\ 898 - 1{,}570\ 796\ 327}{4^2 - 1} \approx 1{,}917\ 807\ 069$$

$$T_{2,2} = \frac{4^2 T_{2,1} - T_{1,1}}{4^2 - 1} = \frac{4^2 \cdot 1{,}917\ 807\ 069 - 2{,}094\ 395\ 102}{4^2 - 1} \approx 1{,}906\ 034\ 533$$